FINITE HORIZON H_∞ CHEAP CONTROL PROBLEM FOR A CLASS OF LINEAR SYSTEMS WITH STATE DELAYS

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ABSTRACT
A finite horizon H-infinity cheap control problem with a given performance level for a linear system with state delays is considered. By a proper transformation of the control variable, this problem is converted to an H-infinity control problem for a singularly perturbed system. For this new problem, parameter-free sufficient conditions of the existence of solution and the solution (controller) itself are obtained. These results are applied to solution of a nonstandard H-infinity control problem.

Keywords: H-infinity cheap control, time-delay system, singular perturbation, nonstandard H-infinity control problem

1. INTRODUCTION

H_∞ control problems are studied extensively in the literature for systems without and with delays in the state variables (Basar and Bernard 1991, Bensoussan, Da Prato, Delfour and Mitter 1992, Doyle, Glover, Khargonekar and Francis 1989, Fridman and Shaked 1998, Fridman and Shaked 2000, van Keulen 1993). The solution of an H_∞ control problem for a linear system can be reduced to a solution of a game-theoretic Riccati equation. In the case of an undelayed system, the Riccati equation is finite dimensional (matrix one), while in the case of a delayed system, it is infinite dimensional (operator one). The operator Riccati equation can be reduced to a hybrid system of three matrix equations of Riccati type. Analysis and solution of this system are very complicated. Therefore, it is extremely important a study of classes of H_∞ control problems with delays, for which the investigation of the operator Riccati equation can be simplified. One of such classes is the class of H_∞ cheap control problems.

The H_∞ cheap control problem is an H_∞ problem with a small control cost (with respect to state and disturbance costs) in the cost functional. A cost functional with a small control cost arises in many topics of control theory. For instance, it arises in the regularization method of a singular optimal control (Bell and Jacobson 1975), in studying the limitations of optimal regulators and filters (Braslavsky, Seron, Maine and Kokotovic 1999, Kwakernaak and Sivan 1972, Seron, Braslavsky, Kokotovic and Mayne 1999), in analysis of control problems with a high control gain (Kokotovic, Khalil and O'Reilly 1986), in the investigation of inverse control problems (Moylan and. Anderson 1973), in the design of a robust control for systems with disturbances (Turetsky and Glizer 2004, Turetsky and Glizer 2011), and some others.

Cheap control problems for systems without disturbances (uncertainties) were widely investigated in the literature. The case of systems with undelayed dynamics was treated more extensively (Bikdash, Nayfeh and Cliff 1993, Jameson and O'Malley 1974/75, Kokotovic, Khalil and O'Reilly 1986, O'Malley and Jameson 1977, Sabery and Sannuti 1987, Smetannikova and Sobolev 2005). The case of systems with delayed dynamics was studied less extensively (Glizer 1999, Glizer 2005, Glizer 2006, Glizer, Fridman and Turetsky 2007, Glizer 2009a). In both cases, an optimal control problem was analyzed. H_∞ cheap control problems have been studied in the literature much less (Glizer 2009b, Toussaint and Basar 2001). It should be noted that two-player zero-sum differential games with a cheap control cost of one of the players in the performance index were analyzed in (Glizer 2000, Petersen 1986, Starr and Ho 1969, Turetsky and Glizer 2004, Turetsky and Glizer 2011). In these works, the case of an undelayed game dynamics and a cheap control cost for the player, minimizing the performance index, was analyzed, which makes the problems, considered in these works, to be close to the H_∞ cheap control problem for a system without delays.

In the present paper, a system with point-wise and distributed state delays and with a square-integrable disturbance is considered. For this system, a finite horizon H_∞ cheap control problem is formulated. A method of asymptotic analysis and solution of the considered H_∞ cheap control problem is proposed. This method is based on: (i) an equivalent transformation of the H_∞ cheap control problem to a new H_∞ problem for a singularly perturbed controlled system; (ii) an asymptotic decomposition of the resulting problem into two much simpler parameter-free subproblems, the slow and fast ones. Using controllers, solving the slow and fast subproblems, a composite controller, solving the transformed problem, is designed. The latter yields a controller, solving the original H_∞ cheap control problem. Note that the algorithm of the analysis of the finite horizon H_∞ cheap control problem, considered in this paper, is similar to that applied in (Glizer 2009b) for analysis of the infinite horizon H_∞ cheap control problem. Along with this, there are essential differences in studying these problems both in main assumptions and techniques.

The results, obtained for the H_∞ cheap control problem, are applied to the solution of a nonstandard H_∞ control problem, i.e. the problem in which the functional does not contain a quadratic control cost.

The following main notations are applied in the paper: (1) E^n is the n-dimensional real Euclidean space; (2) $\|\cdot\|$ denotes the Euclidean norm either of a vector or of a matrix; (3) the prime denotes the transposition of a matrix $A,(A')$ or of a vector $x,(x')$; (4) $L^2[b,c;E^n]$ is the Hilbert space of n-dimensional vector-valued functions square-integrable on the interval $[b,c]$, the norm in this space is denoted as $\|\cdot\|_{L^2[b,c]}$; (5) $C[b,c;E^n]$ is the Banach space of n-dimensional vector-valued functions continuous on the interval $[b,c]$, the norm in this space is denoted as $\|\cdot\|_{C[b,c]}$; (6) I_n is the n-dimensional identity matrix; (7) $\mathrm{col}(x,y)$, where $x \in E^n, y \in E^m$, denotes the column block-vector of the dimension $n+m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x,y) = (x', y')'$.

2. PROBLEM FORMULATION

2.1. H_∞ Cheap Control Problem

Consider the controlled system

$$\frac{dx(t)}{dt} = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\tau)x(t+\tau)d\tau + F_1 w(t), \tag{1}$$

$$\frac{dy(t)}{dt} = A_{21}x(t) + A_{22}y(t) + H_{21}x(t-h) + \int_{-h}^{0} G_{21}(\tau)x(t+\tau)d\tau + Bu(t) + F_2 w(t), \tag{2}$$

where $t \in [0,T]$; $x(t) \in E^n$, $y(t) \in E^m$, $u(t) \in E^m$, (u is a control), $w(t) \in E^q$, (w is a disturbance); $h > 0$ is a given constant time delay; A_{ij}, $(i,j = 1,2)$, H_{i1}, $G_{i1}(\tau)$, F_i, $(i = 1,2)$ and B are given time-invariant matrices of corresponding dimensions; B has the full rank; the matrix-functions $G_{i1}(\tau)$, $(i = 1,2)$ are piece-wise continuous for $\tau \in [-h,0]$.

Assuming that $w(t) \in L^2[0,T;E^q]$, consider the functional

$$J_\varepsilon(u,w) = \int_0^T [x'(t)D_1x(t) + y'(t)D_2y(t) + \varepsilon^2\|u(t)\|^2 - \gamma^2\|w(t)\|^2]dt, \tag{3}$$

where D_1 is symmetric positive-semi-definite, D_2 is symmetric positive-definite matrices; $\gamma > 0$ is a given constant; $\varepsilon > 0$ is a small parameter.

The H_∞ control problem with a performance level γ for the system (1)-(2) is to find a controller $u^*[x(\cdot),y(\cdot)](t)$ that ensures the inequality $J_\varepsilon(u^*,w) \le 0$ along trajectories of (1)-(2) for all $w(t) \in L^2[0,T;E^q]$ and for $x(t) = 0, y(t) = 0, t \le 0.$

The presence of a small multiplier ε^2 in the control cost of the functional (3) means that this problem is the H_∞ cheap control problem.

2.2. Transformation of the Cheap Control Problem

By the control transformation $u(t) = (1/\varepsilon)v(t)$, where v is a new control, the H_∞ problem (1)-(3) becomes

$$\frac{dx(t)}{dt} = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\tau)x(t+\tau)d\tau + F_1 w(t), \tag{4}$$

$$\varepsilon\frac{dy(t)}{dt} = \varepsilon\{A_{21}x(t) + A_{22}y(t) + H_{21}x(t-h) + \int_{-h}^{0} G_{21}(\tau)x(t+\tau)d\tau\} + Bv(t) + \varepsilon F_2 w(t), \tag{5}$$

$$x(t) = 0, \quad y(t) = 0, \quad t \le 0, \tag{6}$$

$$J(v,w) = \int_0^T [x'(t)D_1x(t) + y'(t)D_2y(t) + \|v(t)\|^2 - \gamma^2\|w(t)\|^2]dt. \tag{7}$$

Note that the system (4)-(5) is singularly perturbed (Kokotovic, Khalil and O'Reilly 1986). The state variables $x(\cdot)$ and $y(\cdot)$ are the slow and fast ones, respectively. In this system, the slow state variable is with a delay, while the fast state variable is delay free.

In the sequel, we deal with the H_∞ problem consisting of the system (4)-(5), the initial conditions (6) and the cost functional (7), which is called the original H_∞ control problem (OHICP). Once a controller of the OHICP is obtained, the corresponding controller of the H_∞ problem (1)-(3) is obtained by using the equation $u(t) = (1/\varepsilon)v(t)$.

2.3. Solvability Conditions

Consider the following $(n+m)\times(n+m)$-block-matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad H = \begin{pmatrix} H_{11} & 0 \\ H_{21} & 0 \end{pmatrix}, \tag{8}$$

$$G(\tau) = \begin{pmatrix} G_{11}(\tau) & 0 \\ G_{21}(\tau) & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \tag{9}$$

and the matrix $S(\varepsilon)$

$$\begin{aligned} &S(\varepsilon) = \gamma^{-2}\widetilde{F}\widetilde{F}' - \varepsilon^{-2}\widetilde{B}\widetilde{B}', \\ &\widetilde{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \ \widetilde{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}. \end{aligned} \tag{10}$$

The matrix $S(\varepsilon)$ can be represented in the block form

$$\begin{aligned} S(\varepsilon) &= \begin{pmatrix} \gamma^{-2}F_1F_1' & \gamma^{-2}F_1F_2' \\ \gamma^{-2}F_2F_1' & \gamma^{-2}F_2F_2' - \varepsilon^{-2}BB' \end{pmatrix} \\ &\overset{\Delta}{=} \begin{pmatrix} S_1 & S_2 \\ S_2' & S_3(\varepsilon) \end{pmatrix}. \end{aligned} \tag{11}$$

Consider the following set of Riccati-type ordinary differential and two partial first-order differential equations for the matrices $P(t)$, $Q(t,\tau)$ and $R(t,\tau,\rho)$ in the domain $\Omega =$ $\{(t,\tau,\rho): 0 \le t \le T, -h \le \tau \le 0, -h \le \rho \le 0\}$

$$\begin{aligned} &\frac{dP(t)}{dt} = -P(t)A - A'P(t) \\ &- P(t)S(\varepsilon)P(t) - Q(t,0) - Q'(t,0) - D, \end{aligned} \tag{12}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right)Q(t,\tau) = -(A + S(\varepsilon)P(t))'Q(t,\tau) \\ &-P(t)G(\tau) - R(t,0,\tau) \end{aligned} \tag{13}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \rho}\right)R(t,\tau,\rho) = -G'(\tau)Q(t,\rho) \\ &- Q'(t,\tau)G(\rho) - Q'(t,\tau)S(\varepsilon)Q(t,\rho). \end{aligned} \tag{14}$$

The matrices $P(t)$, $Q(t,\tau)$ and $R(t,\tau,\rho)$ satisfy the boundary conditions

$$P(T) = 0, \ Q(T,\tau) = 0, \ R(T,\tau,\rho) = 0, \tag{15}$$

$$Q(t,-h) = P(t)H, \tag{16}$$

$$\begin{aligned} &R(t,-h,\tau) = H'Q(t,\tau), \\ &R(t,\tau,-h) = Q'(t,\tau)H. \end{aligned} \tag{17}$$

It is seen that the matrix-functions $Q(t,\tau)$ and $R(t,\tau,\rho)$ are present in the set (12)-(14) with deviating arguments. The problem (12)-(17) is, in general, of a high dimension. Moreover, due to the expression for $S(\varepsilon)$, this problem is ill-posed for $\varepsilon \to +0$.

Lemma 2.1. *Let, for a given* $\varepsilon > 0$, *there exist a solution* $\{P(t,\varepsilon), Q(t,\tau,\varepsilon), R(t,\tau,\rho,\varepsilon)\}$ *of (12)-(17) such that*

$$\begin{aligned} &P'(t,\varepsilon) = P(t,\varepsilon), \\ &R'(t,\tau,\rho,\varepsilon) = R(t,\rho,\tau,\varepsilon). \end{aligned} \tag{18}$$

Then, for this ε, *the controller*

$$\begin{aligned} &v^*[x(\cdot),y(\cdot)](t) = -\varepsilon^{-1}B'[P(t,\varepsilon)z(t) \\ &+ \int_{-h}^{0} Q(t,\tau,\varepsilon)z(t+\tau)d\tau], \ z = col(x,y) \end{aligned} \tag{19}$$

solves the OHICP.

Proof. The lemma is a direct technical extension of the result of (Glizer 2003) (Theorem 2.1 and its proof) where the case of only a point-wise state delay in the controlled system has been considered.

Remark 2.1. Due to the above mentioned features of the problem (12)-(17), verifying the existence of its solution and searching this solution are very complicated tasks.

2.4. Objectives of the Paper

The objectives of this paper are the following:

(i) to derive ε-free reduced conditions, guaranteeing the existence of a controller solving the OHICP for all sufficiently small $\varepsilon > 0$;

(ii) to design a controller much simpler than (19), which being ε-free solves the OHICP for all sufficiently small $\varepsilon > 0$;

(iii) to apply the above mentioned results to the solution of a nonstandard H_∞ control problem.

3. FORMAL CONSTRUCTING A SIMPLIFIED CONTROLLER FOR THE OHICP

In this section, we propose a method of constructing a simplified controller for the OHICP. This method is based on an asymptotic decomposition of the OHICP into two much simpler ε-free subproblems, the slow and fast ones.

3.1. Slow Subproblem

The slow subproblem is obtained from the OHICP by setting there formally $\varepsilon = 0$ and redenoting x, y, v and J by x_s, y_s, v_s and J_s, respectively. Thus, we have

$$\frac{dx_s(t)}{dt} = A_{11}x_s(t) + A_{12}y_s(t) + H_{11}x_s(t-h)$$

$$+\int_{-h}^{0} G_{11}(\tau)x_s(t+\tau)d\tau + F_1 w(t), \quad t > 0, \tag{20}$$

$$Bv_s(t) = 0, \quad t \in [0, +\infty), \tag{21}$$

$$x_s(t) = 0, \quad t \le 0, \tag{22}$$

$$J_s = \int_0^T [x_s'(t)D_1 x_s(t) + y_s'(t)D_2 y_s(t) + \|v_s(t)\|^2 - \gamma^2 \|w(t)\|^2]dt. \tag{23}$$

Since the matrix B is invertible, the equation (21) yields

$$v_s(t) = 0, \quad t \in [0, +\infty). \tag{24}$$

By substituting (24) into (23), we obtain

$$J_s = \int_0^T [x_s'(t)D_1 x_s(t) + y_s'(t)D_2 y_s(t) - \gamma^2 \|w(t)\|^2]dt. \tag{25}$$

Since the variable $y_s(t)$ does not satisfy any equation for $t \in [0, +\infty)$, one can choose it to satisfy a desirable property of the system (20). This means that the variable $y_s(t)$ can be considered as a control variable in the system (20). Thus, the functional (25), calculated along trajectories of this system, depends on the control variable $y_s(t)$ and the disturbance $w(t) \in L^2[0, T; E^q]$, i.e., $J_s = J_s(y_s, w)$. For the system (20), we can formulate the following H_∞ control problem with a performance level γ: to find a controller $y_s[x_s(\cdot)](t)$ that ensures the inequality $J_s(y_s, w) \le 0$ along trajectories of (20),(22) for all $w(t) \in L^2[0, T; E^q]$. This H_∞ control problem is called the slow H_∞ control subproblem (SHICP) associated with the OHICP.

Consider the following set of Riccati-type matrix ordinary differential and two first-order partial differential equations with deviating arguments:

$$\frac{dP_s(t)}{dt} = -P_s(t)A_{11} - A_{11}'P_s(t) - P_s(t)S_s P_s(t) - Q_s(t,0) - Q_s'(t,0) - D_1, \tag{26}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right)Q_s(t,\tau) = -(A_{11}' + P_s(t)S_s)Q_s(t,\tau) - P_s(t)G_{11}(\tau) - R_s(t,0,\tau), \tag{27}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \rho}\right)R_s(t,\tau,\rho) = -G_1'(\tau)Q_s(t,\rho) - Q_s'(t,\tau)G_1(\rho) - Q_s'(t,\tau)S_s Q_s(t,\rho), \tag{28}$$

where $S_s = \gamma^{-2}F_1F_1' - A_{12}D_2^{-1}A_{12}'$.

The set of equations (26)-(28) is subject to the boundary conditions

$$P_s(T) = 0, \quad Q_s(T,\tau) = 0, \quad R_s(T,\tau,\rho) = 0, \tag{29}$$

$$Q_s(t,-h) = P_s(t)H_{11}, \tag{30}$$

$$R_s(t,-h,\tau) = H_{11}'Q_s(t,\tau), \quad R_s(t,\tau,-h) = Q_s'(t,\tau)H_{11}. \tag{31}$$

Similarly to Lemma 2.1, one has the following proposition.

Proposition 3.1. *Let there exist a solution* $\{P_s(t), Q_s(t,\tau), R_s(t,\tau,\rho)\}$ *of (26)-(31) in the domain* $\boldsymbol{\Omega}$*, such that*

$$P_s'(t) = P_s(t), \quad R_s'(t,\tau,\rho) = R_s(t,\rho,\tau). \tag{32}$$

Then, the controller

$$y_s^*[x_s(\cdot)](t) = -D_2^{-1}A_{12}' \times \left[P_s x_s(t) + \int_{-h}^{0} Q_s(\tau) x_s(t+\tau) d\tau \right] \quad (33)$$

solves the SHICP.

3.2. Fast Subproblem

The fast subproblem is obtained in the following three stages. First, the slow variable $x(\cdot)$ is removed from the equation (5) and the cost functional (7) of the OHICP. Second, the following transformation of variables is made in the resulting problem:

$$t = \varepsilon\xi, \quad y(\varepsilon\xi) = y_f(\xi), \quad v(\varepsilon\xi) = v_f(\xi), \quad w(\varepsilon\xi) = w_f(\xi), \quad J(v(\varepsilon\xi), w(\varepsilon\xi)) = \varepsilon J_{f,1}(v_f(\xi), w_f(\xi)), \quad (34)$$

where ξ, y_f, v_f, w_f and J_f are new independent variable, state, control, disturbance and cost functional, respectively. Thus, we obtain the system and the cost functional

$$dy_f(\xi)/d\xi = \varepsilon A_{22} y_f(\xi) + Bv_f(\xi) + \varepsilon F_2 w_f(\xi), \ \ \xi > 0, \quad (35)$$

$$J_f(v_f, w_f) = \int_0^{T/\varepsilon} [\, y_f'(\xi) D_2 y_f(\xi) + v_f'(\xi) v_f(\xi) - \gamma^2 \|w_f(\xi)\|^2 \,] d\xi. \quad (36)$$

Finally, neglecting formally the terms with the multiplier ε in (35) and replacing formally T/ε by $+\infty$ in (36) yields the system

$$dy_f(\xi)/d\xi = Bv_f(\xi), \quad \xi > 0. \quad (37)$$

and the cost functional

$$J_f(v_f, w_f) = \int_0^{+\infty} [\, y_f'(\xi) D_2 y_f(\xi) + v_f'(\xi) v_f(\xi) - \gamma^2 \|w_f(\xi)\|^2 \,] d\xi. \quad (38)$$

For the system (37) and the cost functional (38), the H_∞ control problem with a performance level γ can be formulated as follows. To find a controller $v_f^*[y_f(\xi)]$ that stabilizes (37) and ensures the inequality $J_f(v_f^*, w_f) \le 0$ along its trajectories for all $w_f(\xi) \in L_2[0,+\infty; E^q]$ and for $y_f(0) = 0$. This H_∞ control problem is called the fast H_∞ control subproblem (FHICP) associated with the OHICP.

Let K be any $m \times m$ -matrix such that BK is a Hurwitz matrix. Then, the controller

$$v_f^*[y_f(\xi)] = Ky_f(\xi) \quad (39)$$

solves the FHICP.

Note, that the FHICP is a particular case of the infinite horizon H_∞ control problem, considered in (Basar and Bernard 1991). Due to results of this book, if there exists a solution P_f of the algebraic matrix Riccati-type equation

$$-P_f BB' P_f + D_2 = 0, \quad (40)$$

such that $-BB'P_f$ is a Hurwitz matrix, then the matrix gain K in (39) can be chosen as

$$K = -B' P_f. \quad (41)$$

Let us show that the above mentioned solution of (40) exists. Indeed, since the matrix D_2 is positive definite, then there exist the unique positive definite solution of (40)

$$P_f = (BB')^{-1/2} \times ((BB')^{1/2} D_2 (BB')^{1/2})^{1/2} (BB')^{-1/2}, \quad (42)$$

where the superscript "1/2" denotes the unique symmetric positive definite square root of respective symmetric positive definite matrix, the one "-1/2" denotes the square root of respective inverse matrix.
Now, using (41) and (42), we have

$$BK = -(BB')^{1/2} \times ((BB')^{1/2} D_2 (BB')^{1/2})^{1/2} (BB')^{-1/2}. \quad (43)$$

Since D_2 is positive definite and B is not singular, then the equation (43) means that the matrix BK is Hurwitz. Hence, the controller

$$v_f^*[y_f(\xi)] = -B' P_f y_f(\xi) \quad (44)$$

solves the FHICP.

3.3. Composite Controller for the OHICP

In this subsection, based on the control $v_s(t)$, given by (24), the controller $y_s^*[x_s(\cdot)](t)$, solving the SHICP, and the controller $v_f^*[y_f(\xi)]$, solving the FHICP, we construct a composite controller for the OHICP. Then, we show that this controller solves the OHICP for all sufficiently small $\varepsilon > 0$.

The composite controller is constructed in the form

$$v_c[x(\cdot), y(\cdot)](t) = v_s(t) + v_f^*[\tilde{y}(t/\varepsilon)], \quad (45)$$

where $\tilde{y}(t/\varepsilon)$ is defined as follows

$$\tilde{y}(t/\varepsilon) \stackrel{\Delta}{=} y(t) - y_s^*[x(\cdot)](t). \quad (46)$$

Substituting (24) and (44) into (45), and using (33), (42) and (46) yield after some rearrangement

$$v_c[x(\cdot), y(\cdot)](t) = -B' P_f [D_2^{-1} A_{12}' P_s(t) x(t) + y(t) + \int_{-h}^{0} D_2^{-1} A_{12}' Q_s(t,\tau) x(t+\tau) d\tau] \quad (47)$$

4. ε-FREE SOLVABILITY CONDITIONS FOR THE OHICP

Theorem 4.1. *Let there exist a solution* $\{P_s(t), Q_s(t,\tau), R_s(t,\tau,\rho)\}$ *of (26)-(31) in the domain* $\boldsymbol{\Omega}$*, satisfying the conditions (32). Then, there exists a positive number* ε^**, such that the controller (47) solves the OHICP for all* $\varepsilon \in (0, \varepsilon^*]$.

Proof. The proof of the theorem consists of four parts. For the sake of saving the space of the paper, we present here a sketch of the proof.

Part I. By substituting the controller (47) into the system (4)-(5) and the cost functional (7), we obtain

$$\frac{dx(t)}{dt} = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\tau)x(t+\tau)d\tau + F_1 w(t), \quad (48)$$

$$\varepsilon\frac{dy(t)}{dt} = [\varepsilon A_{21} - BB' P_f D_2^{-1} A_{12}' P_s(t)]x(t) + [\varepsilon A_{22} - BB' P_f]y(t) + \varepsilon H_{21}x(t-h) + \int_{-h}^{0}\left[\varepsilon G_{21}(\tau) - BB' P_f D_2^{-1} A_{12}' Q_s(t,\tau)\right]\times x(t+\tau)d\tau + \varepsilon F_2 w(t), \quad (49)$$

$$J(\bar{v}_c, w) \stackrel{\Delta}{=} J_c(w) = \int_0^T \Big[x'(t)D_{P1}(t)x(t) + 2x'(t)D_{P2}(t)y(t) + y'(t)D_{P3}y(t) + 2x'(t)\int_{-h}^{0} D_{Q1}(t,\tau)x(t+\tau)d\tau + 2y'(t)\int_{-h}^{0} D_{Q2}(t,\tau)x(t+\tau)d\tau + \int_{-h}^{0}\int_{-h}^{0} x'(t+\tau)D_{R1}(t,\tau,\rho)x(t+\rho)d\tau d\rho - \gamma^2\|w(t)\|^2\Big]dt, \quad (50)$$

where

$$D_{P1}(t) = D_1 + P_s(t)A_{12}D_2^{-1}A_{12}'P_s(t), \quad (51)$$

$$D_{P2}(t) = P_s(t)A_{12}, \; D_{P3} = 2D_2, \quad (52)$$

$$D_{Q1}(t,\tau) = P_s(t)A_{12}D_2^{-1}A_{12}'Q_s(t,\tau), \quad D_{Q2}(t,\tau) = A_{12}'Q_s(t,\tau), \quad (53)$$

$$D_{R1}(t,\tau,\rho) = Q_s'(t,\tau)A_{12}D_2^{-1}A_{12}'Q_s(t,\rho). \quad (54)$$

Thus, the proof of the theorem is reduced to a proof of fulfilment of the following inequality for all sufficiently small $\varepsilon > 0$:

$$J_c(w) \le 0 \quad \forall w(\cdot) \in L^2[0,T;E^q], \quad (55)$$

along trajectories of the system (48)-(49) subject to the initial conditions (6).

Part II. Consider the following $(n+m)\times(n+m)$ block matrices

$$\hat{A}(t,\varepsilon) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} - \varepsilon^{-1}P_f^{-1}A_{12}'P_s(t) & A_{22} - \varepsilon^{-1}BB'P_f \end{pmatrix}, \quad (56)$$

$$\hat{G}(t,\tau,\varepsilon) = \begin{pmatrix} G_{11}(\tau) & 0 \\ G_{21}(\tau) - \varepsilon^{-1}P_f^{-1}A_{12}'Q_s(t,\tau) & 0 \end{pmatrix}, \quad (57)$$

$$D_P(t) = \begin{pmatrix} D_{P1}(t) & D_{P2}(t) \\ D_{P2}'(t) & D_{P3}(t) \end{pmatrix}, \quad D_Q(t,\tau) = \begin{pmatrix} D_{Q1}(t,\tau) & 0 \\ D_{Q2}(t,\tau) & 0 \end{pmatrix}, \quad (58)$$

$$D_R(t,\tau,\rho) = \begin{pmatrix} D_{R1}(t,\tau,\rho) & 0 \\ 0 & 0 \end{pmatrix}, \quad S_F = \gamma^{-2}FF'. \quad (59)$$

Consider the following system of ordinary and partial matrix differential equations of Riccati type with respect to $(n+m)\times(n+m)$-matrices $\hat{P}(t)$, $\hat{Q}(t,\tau)$ and $\hat{R}(t,\tau,\rho)$ in the domain Ω:

$$\frac{d\hat{P}(t)}{dt} = -\hat{P}(t)\hat{A}(t,\varepsilon) - \hat{A}'(t,\varepsilon)\hat{P}(t) - \hat{P}(t)S_F\hat{P}(t) - \hat{Q}(t,0) - \hat{Q}'(t,0) - D_P(t), \quad (60)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right)\hat{Q}(t,\tau) = -(\hat{A}(t,\varepsilon) + S_F\hat{P}(t))'\hat{Q}(t,\tau) - \hat{P}(t)\hat{G}(t,\tau,\varepsilon) - \hat{R}(t,0,\tau) - D_Q(t,\tau), \quad (61)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \rho}\right)\hat{R}(t,\tau,\rho) = -\hat{G}'(t,\tau,\varepsilon)\hat{Q}(t,\rho) - \hat{Q}'(t,\tau)\hat{G}(t,\rho,\varepsilon) - \hat{Q}'(t,\tau)S_F\hat{Q}(t,\rho) - D_R(t,\tau,\rho). \quad (62)$$

The system (60)-(62) is considered subject to the boundary conditions

$$\hat{P}(T) = 0, \quad \hat{Q}(T,\tau) = 0, \quad \hat{R}(T,\tau,\rho) = 0, \quad (63)$$

$$\hat{Q}(t,-h) = \hat{P}(t)H, \quad \hat{R}(t,-h,\tau) = H'\hat{Q}(t,\tau), \quad \hat{R}(t,\tau,-h) = \hat{Q}'(t,\tau)H, \quad (64)$$

where the matrix H is given in (8).

Let us show the following. If for some $\varepsilon > 0$, the problem (60)-(64) has a solution

$\{\hat{P}(t,\varepsilon), \hat{Q}(t,\tau,\varepsilon), \hat{R}(t,\tau,\rho,\varepsilon)\}$ such that

$$\hat{P}'(t,\varepsilon) = \hat{P}(t,\varepsilon), \quad \hat{R}'(t,\tau,\rho,\varepsilon) = \hat{R}(t,\rho,\tau,\varepsilon), \quad (t,\tau,\rho) \in \Omega, \quad (65)$$

then, for this ε, the inequality (55) is satisfied along trajectories of the system (48)-(49) subject to the initial conditions (6).

Part III. Consider the following functional, depending on the parameter $t \in [0,T]$, on a vector $\varphi_0 \in E^{n+m}$ and on a function $\varphi_z(\theta) \in L^2[t-h,t;E^{n+m}]$:

$$V[t,\varphi_0,\varphi_z(\theta)] \stackrel{\Delta}{=} \varphi_0'\hat{P}(t,\varepsilon)\varphi_0 + 2\varphi_0'\int_{t-h}^{t}\hat{Q}(t,\theta-t,\varepsilon)\varphi_z(\theta)d\theta + \int_{t-h}^{t}\int_{t-h}^{t}\varphi_z'(\theta)\hat{R}(t,\theta-t,\sigma-t,\varepsilon)\times \varphi_z(\sigma)d\theta d\sigma. \quad (66)$$

Let, for a given $w(\cdot) \in L^2[0,T;E^q]$, the vector function $\hat{z}[t,w(\cdot)] = col(\hat{x}[t,w(t)], \hat{y}[t,w(\cdot)])$, $t \in [0,T]$, be the solution of the system (48)-(49) subject to the initial conditions (6). Such a solution exists and is unique. Let, for a given $t \in [0,T]$, $\hat{V}(t) \stackrel{\Delta}{=} V[t,\hat{z}[t,w(\cdot)],\hat{z}[\theta,w(\cdot)]]$. Calculating the derivative of $\hat{V}(t)$, we obtain after some rearrangement

$$\frac{d\hat{V}(t)}{dt} = -\hat{z}'[t,w(\cdot)]D_P(t)\hat{z}[t,w(\cdot)] - 2\hat{z}'[t,w(\cdot)]\int_{t-h}^{t}D_Q(t,\theta-t)\hat{z}[\theta,w(\cdot)]d\theta - \int_{t-h}^{t}\int_{t-h}^{t}\hat{z}'[\theta,w(\cdot)]D_R(t,\theta-t,\sigma-t)\times \hat{z}[\sigma,w(\cdot)]d\theta d\sigma + \gamma^2\|w(t)\|^2 - \gamma^2\|w(t) - \hat{w}[t,w(\cdot)]\|^2, \quad (67)$$

where

$$\hat{w}[t,w(\cdot)] = \gamma^{-2}F'[\hat{P}(t,\varepsilon)\bar{z}[t,w(\cdot)] + \int_{t-h}^{t}\hat{Q}(t,\theta-t,\varepsilon)\bar{z}[\theta,w(\cdot)]d\theta]. \quad (68)$$

Equation (67) directly yields the inequality $\forall t \in [0,T]$

$$\frac{d\hat{V}(t)}{dt} + \hat{z}'[t,w(\cdot)]D_P(t)\hat{z}[t,w(\cdot)] + 2\hat{z}'[t,w(\cdot)]\int_{t-h}^{t}D_Q(t,\theta-t)\hat{z}[\theta,w(\cdot)]d\theta + \int_{t-h}^{t}\int_{t-h}^{t}\hat{z}'[\theta,w(\cdot)]D_R(t,\theta-t,\sigma-t)\times \hat{z}[\sigma,w(\cdot)]d\theta d\sigma - \gamma^2\|w(t)\|^2 \le 0. \quad (69)$$

Integrating the inequality (69) from $t = 0$ to $t = T$ and using the conditions (6) and (63) immediately yield the fulfilment of the inequality (55) along trajectories of the system (48)-(49) subject to the initial conditions (6).

Part IV. The existence of solution to the problem (60)-(64) for all sufficiently small $\varepsilon > 0$, which satisfies the conditions (65), is shown by formal constructing and justifying the zero-order asymptotic solution to the problem (60)-(64). This asymptotic solution can be obtained in the way similar (but not identical) to (Glizer 1999). This completes the proof of the theorem.

Corollary 4.1. *Under the conditions of Theorem 4.1, the controller*

$$u_c[x(\cdot),y(\cdot)](t)=(1/\varepsilon)v_c[x(\cdot),y(\cdot)](t)$$

solves the H_∞ *control problem (1)-(3) for all* $\varepsilon\in(0,\varepsilon^*]$.

5. NONSTANDARD H_∞ CONTROL PROBLEM

In this section, we consider the following functional for the system (1)-(2):

$$J(u,w)=\int_0^T[x'(t)D_1x(t)+y'(t)D_2y(t) -\gamma^2\|w(t)\|^2]dt. \tag{70}$$

It is seen that the functional (70) does not contain a quadratic control cost.

The nonstandard H_∞ control problem with a performance level γ for the system (1)-(2) (NHICP) is to find a controller $u^*[x(\cdot),y(\cdot)](t)$ that ensures the inequality $J(u^*,w)\le 0$ along trajectories of (1)-(2) for all $w(t)\in L^2[0,T;E^q]$ and for $x(t)=0,\ y(t)=0,\ t\le 0.$

Since the functional (70) does not contain a quadratic control cost, the approach, proposed in Lemma 2.1 for the solution of the H_∞ control problem (4)-(7), is not applicable for the solution of the NHICP. In order to solve the NHICP, we replace the functional (70) with the cheap control functional (3). Such a replacing leads to the H_∞ control problem (4)-(7), for which Corollary 4.1 gives ε-free reduced-order solvability conditions, as well as the controller solving this problem. Due to this corollary, the employing the controller $u(t)=u_c[(x(\cdot),y(\cdot)](t)$ in the system (1)-(2) subject to the initial conditions $x(t)=0,\ y(t)=0,\ t\le 0$ yields the following inequality for all $w(t)\in L^2[0,T;E^q]$ and all $\varepsilon\in(0,\varepsilon^*]$:

$$J(u_c,w)+\int_0^T\|v_c[x(\cdot),y(\cdot)](t)\|^2dt\le 0, \tag{71}$$

where the arguments $x(\cdot)$ and $y(\cdot)$ for v_c constitute the solution $col(x(t,\varepsilon),y(t,\varepsilon))$ of the system (48)-(49) subject to the initial conditions (6).

From the inequality (71), one directly has $J(u_c,w)\le 0$, which implies that the controller $u(t)=u_c[(x(\cdot),y(\cdot)](t)$ solves the NHICP for all $\varepsilon\in(0,\varepsilon^*]$ if there exist a solution $\{P_s(t),Q_s(t,\tau),R_s(t,\tau,\rho)\}$ of (26)-(31) in the domain Ω, satisfying the conditions (32).

Remark 5.1. The inequality (71) yields a stronger inequality than $J(u_c,w)\le 0$. Namely,

$$J(u_c,w)\le-\int_0^T\|v_c[x(\cdot,\varepsilon),y(\cdot,\varepsilon)](t)\|^2dt. \tag{72}$$

The integral in the right-hand side of this inequality depends on $\varepsilon\in(0,\varepsilon^*]$. The following theorem gives an estimate of this integral for small enough $\varepsilon>0$.

Theorem 5.1. *Let there exist a solution* $\{P_s(t),Q_s(t,\tau),R_s(t,\tau,\rho)\}$ *of (26)-(31) in the domain* Ω*, satisfying the conditions (32). Then there exists a positive number* ε_1^*, $(\varepsilon_1^*\le\varepsilon^*)$*, such that, for any given* $w(t)\in L^2[0,T;E^q]$ *and all* $\varepsilon\in(0,\varepsilon_1^*]$*, the following inequality is satisfied:*

$$0\le\int_0^T\|v_c[x(\cdot,\varepsilon),y(\cdot,\varepsilon)](t)\|^2dt\le a\varepsilon\left(\|w(t)\|_{L^2[0,T]}\right)^2, \tag{73}$$

where $a>0$ *is some constant independent of* ε.

Proof. In order to save the space, we present here a sketch of the proof.

The left-hand inequality in (73) is obvious. Proceed to the proof of the right-hand one.

Asymptotic analysis of the problem (48)-(49), (6) leads to the existence of a constant $0<\varepsilon_1^*\le\varepsilon^*$ such that, for any given $w(t)\in L^2[0,T;E^q]$ and all $\varepsilon\in(0,\varepsilon_1^*]$, the following inequalities are valid:

$$\|x(t,\varepsilon)-\bar{x}(t)\|_{C[0,T]}\le a_1\varepsilon\|w(t)\|_{L^2[0,T]}, \tag{74}$$

$$\|y(t,\varepsilon)-\bar{y}(t)\|_{C[0,T]}\le a_1\varepsilon^{1/2}\|w(t)\|_{L^2[0,T]}, \tag{75}$$

where $a_1>0$ is some constant independent of ε,

$$\bar{x}(t)=\int_0^t\overline{\Phi}_x(t,s)F_1w(s)ds, \tag{76}$$

$$\bar{y}(t)=\int_0^t\overline{\Phi}_y(t,s)F_1w(s)ds, \tag{77}$$

the $n\times n$-matrix-valued function $\overline{\Phi}_x(t,s)$ is the solution of the following problem for $0\le s\le t\le T$:

$$\frac{d\overline{\Phi}_x(t,s)}{dt}=\left(A_{11}-A_{12}D_2^{-1}A_{12}'P_s(t)\right)\overline{\Phi}_x(t,s) +H_{11}\overline{\Phi}(t-h,s)+\int_{-h}^0\big(G_{11}(\tau)$$

$$-A_{12}D_2^{-1}A_{12}^{'}Q_s(t,\tau)\Big)\overline{\Phi}_x(t+\tau,s)ds, \qquad (78)$$

$$\overline{\Phi}_x(t,s)=0, \quad t<s; \quad \overline{\Phi}_x(s,s)=I_n, \qquad (79)$$

and the $m \times n$ -matrix-valued function $\overline{\Phi}_y(t,s)$ has the form

$$\overline{\Phi}_y(t,s)=-D_2^{-1}A_{12}^{'}P_s(t)\overline{\Phi}_x(t,s)$$

$$-\int_{-h}^{0}D_2^{-1}A_{12}^{'}Q_s(t,\tau)\overline{\Phi}_x(t+\tau,s)d\tau. \qquad (80)$$

Now, by using the inequalities (74)-(75) and the equations (76)-(80), one obtains after some rearrangement the inequality

$$\begin{aligned}&\|v_c[x(\cdot,\varepsilon),y(\cdot,\varepsilon)](t)\|_{C[0,T]}\le\\&a_2\varepsilon^{1/2}\|w(t)\|_{L^2[0,T]},\end{aligned} \qquad (81)$$

where $a_2>0$ is some constant independent of ε.

The inequality (81) directly yields the right-hand inequality in (73).

6. CONCLUSIONS

In this paper, a linear controlled system with point-wise and distributed state delays and a square-integrable disturbance is considered. For the sake of simplicity, it is assumed that this system consists of two modes. One of them is controlled directly, while the other is controlled through the first one. Moreover, it is considered the case where the state variable of the directly controlled mode has no delays. For this system, the finite horizon H_∞ control problem with a given performance level is studied. The control cost in the cost functional of this problem is assumed to be small with respect to the state and disturbance costs, i.e., the considered problem is the H_∞ cheap control problem. By using a simple control transformation, this problem is converted to the H_∞ control problem for a system with a small multiplier $\varepsilon>0$ for a part of the derivatives, i.e., for a singularly perturbed system. In this singularly perturbed system, the slow state variable has delays, while the fast state variable has not. This new H_∞ control problem, considered as an original one, is analyzed in the sequel of the paper. For this problem, reduced-order solvability conditions, valid for any positive small enough ε, are derived. The ε-free controller, solving this problem for all sufficiently small values $\varepsilon>0$, also is designed. This controller, being multiplied by $1/\varepsilon$, yields the controller, solving the H_∞ cheap control problem. These results are applied to the solution of the nonstandard H_∞ control problem. It is shown that the controller, solving the H_∞ cheap control problem, also solves the nonstandard H_∞ control problem. Moreover, it is shown that this controller ensures the cost functional of the nonstandard H_∞ control problem to be smaller than the negative function of ε, the absolute value of which is of order of ε for all sufficiently small values $\varepsilon>0$.

ACKNOWLEDGMENTS

This research was partially supported by KAMEA fund (Ministry of Immigrant Absorption, State of Israel).

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