ROBUST CONTROLLABILITY SETS OF LINEAR AND SATURATED LINEAR STRATEGIES WITH DISCONTINUOUS GAINS

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ABSTRACT
A linear system with scalar control and disturbance is considered. The robust controllability of this system to a given target set is studied in the classes of linear and saturated linear strategies. In previous works, the analysis of this problem was limited to continuous sign-constant gains. Such an approach has been inspired by those practical problems, where the control coefficient function in the scalar system, corresponding to the original one, is sign-constant. For systems of sign-varying control coefficient functions using a sign-varying discontinuous gain is proposed. It is shown that using such a gain considerably increases the robust controllability set of the corresponding strategy.

Keywords: linear controlled system, disturbance, linear feedback strategy, saturated linear feedback strategy, robust controllability set

1. INTRODUCTION

Controllability is one of the basic system properties. This property has been well studied for systems without uncertainties by using an open-loop control (Kalman, 1960; Kwakernaak and Sivan, 1972; Bryson and Ho, 1975; Gabasov and Kirillova, 1976). However, this elegant theory is not applicable to real-life systems, affected by unmeasurable input parameters (uncertainties). For such systems, controllability should be augmented by the robustness property with respect to any admissible uncertainty realization. As a rule, the robust controllability can be realized by a feedback control. In the framework of differential games of kind, this property (called sometimes playability) was studied extensively (Isaacs, 1965; Blaquiere et al., 1969; Krasovskii and Subbotin, 1988; Lewin, 1994). There, the input uncertainty (disturbance) is considered as the control of an opponent. Various types of robust feedback controllability of systems with uncertainties were investigated in (Petersen et al., 1992; Savkin, 1997; Savkin and Petersen, 1999; Turetsky and Glizer, 2004; Ganebny et al., 2006).

The general robust controllability concept does not take into account possible control constraints, although such constraints are indispensable part of most practical control problems. For analysis of the robust controllability problem with control constraints, in (Glizer and Turetsky, 2012), it was developed a concept of a *robust controllability set*. According to this concept, the robust controllability set is constructed for a so-called robust transferring feedback strategy, which steers the closed-loop system from the maximal possible set of initial positions to a prescribed target set against any admissible disturbance. The time realizations of such a strategy may violate the prescribed hard control constraints for some initial positions and for some admissible disturbances. By taking into account the hard constraints, the maximal possible set of initial positions is reduced to the robust controllability set. If the system trajectory emanates from any point of this set, the corresponding time realization of the robust transferring strategy satisfes the control constraint, robustly with respect to all admissible disturbances.

In (Glizer and Turetsky, 2012), two classes of robust transferring strategies – linear and saturated linear – have been studied. The gains of these strategies were assumed to be non-zero and smooth, which means that they are sign-constant. Such gains are effective in the case where the control coefficient function in the scalar system, corresponding to the original one, is sign-constant. However, a non-minimum phase controller, which can be found in some applications, leads to a sign- varying control coefficient function. In this case, the robust controllability set of a strategy with a sign-constant gain becomes small or even empty. In this paper, we propose to use non-zero sign-varying gains with the sign opposite to the sign of the control coefficient function. Note that such gains are necessarily discontinuous. It is shown that such extension of the class of admissible gains enlarges considerably the robust controllability sets of linear strategies and of saturated linear strategies.

2. PROBLEM STATEMENT

2.1. Original Controlled System

Consider a controlled dynamic system

$$\dot{x} = A(t)x + b(t)u + c(t)v + f(t), \quad 0 \le t \le t_f, \quad (1)$$

where $x \in R^n$ is a state vector; $u \in R^1$ and $v \in R^1$ are the control and the disturbance, respectively; t_f is a prescribed final instant of time; the matrix-valued function $A(t)$ and the vector-valued functions $b(t)$, $c(t)$, $f(t)$ are continuous on $[0, t_f]$.

The control and the disturbance satisfy the constraints

$$|u| \le \rho_u, \quad (2)$$

$$|v| \le \rho_v, \quad (3)$$

where ρ_u, ρ_v are given positive constants.

Definition 1 *A function* $u = u(t,x)$, $(t,x) \in \mathcal{S}_x \triangleq \{(t,x): t \in [0, t_f), x \in R^n\}$, *is called an admissible feedback strategy for the system (1), if the corresponding closed-loop system has a unique absolutely continuous solution* $x(t)$, $t \in [0, t_f)$, *for any admissible disturbance* $v(t)$ *and for any initial condition* $x(t_0) = x_0$, $(t_0, x_0) \in \mathcal{S}_x$. *It is also assumed that there exists*

$$x(t_f) \triangleq \lim_{t \to t_f - 0} x(t). \quad (4)$$

The target set is the linear manifold in (t,x)-space

$$\mathcal{T}_x = \{(t,x): t = t_f, d^T x + d_0 = 0\}, \quad (5)$$

where $d = (d_1, d_2, \ldots, d_n)^T \in R^n$ is a prescribed non-zero vector, d_0 is a prescribed scalar.

The control objective is to bring the system (1) from a given initial position $x(t_0) = x_0$, $(t_0, x_0) \in \mathcal{S}_x$, to the target set (5), respecting the control constraint (2), by means of an admissible feedback strategy $u(t,x)$, for all admissible disturbances $v(t)$.

Definition 2 *An admissible strategy is called robust transferring from a given initial position* $(t_0, x_0) \in \mathcal{S}_x$ *to* $\mathcal{T}_x$, *if for any admissible* $v(t)$: $x(t_f) \in \mathcal{T}_x$. *It is called robust transferring from* $\mathcal{M}_x \subseteq \mathcal{S}_x$ *to* $\mathcal{T}_x$, *if it is robust transferring from any point* $(t_0, x_0) \in \mathcal{M}_x$ *to* $\mathcal{T}_x$.

Let, for a given admissible strategy, the set $\mathcal{M}_x^{\max} \subseteq \mathcal{S}_x$ be the maximal set, from which it is robust transferring to $\mathcal{T}_x$. The set $\mathcal{M}_x^{\max} = \mathcal{M}_x^{\max}(u(\cdot))$ is called the *robust transferrable set* of the strategy $u(\cdot)$.

Definition 3 *The subset* $\mathcal{C}_x = \mathcal{C}_x(u(\cdot))$ *of the robust transferrable set* $\mathcal{M}_x^{\max}$ *is called the robust controllability set of* $u(\cdot)$, *if:*

(i) *for any initial point* $(t_0, x_0) \in \mathcal{C}_x$ *and any admissible disturbance, the time realization of* $u(t,x)$ *along the trajectory* $x = x(t)$ *satisfies the control constraint (2):*

$$|u(t, x(t))| \le \rho_u, \quad t \in [t_0, t_f). \quad (6)$$

(ii) *for any initial point* $(t_0, x_0) \in \mathcal{M}_x^{\max} \setminus \mathcal{C}_x$ *there exist an admissible disturbance and a time moment* $t_1 \in [t_0, t_f)$, *such that*

$$|u(t_1, x(t_1))| > \rho_u. \quad (7)$$

2.2. Scalarization

Let $\Phi(t,\tau)$, $0 \le \tau \le t \le t_f$, be the fundamental matrix of the homogeneous system corresponding to (1). By the non-homogenous transformation of the state variable in (1),

$$z = z(t,x) = d^T \left(\Phi(t_f, t)x + \int_t^{t_f} \Phi(t_f, \tau) f(\tau) d\tau \right) + d_0, \quad (8)$$

this system is reduced (Glizer and Turetsky, 2012; Gutman, 2006) to the scalar equation

$$\dot{z} = h_1(t)u + h_2(t)v, \quad (9)$$

where

$$h_1(t) = d^T \Phi(t_f, t) b(t), \qquad h_2(t) = d^T \Phi(t_f, t) c(t). \quad (10)$$

Note that due to the continuity of $A(t)$, $b(t)$ and $c(t)$, the functions $h_1(t)$ and $h_2(t)$ are continuous on $[0, t_f]$. For the scalar system (9), the target set (5) becomes

$$\mathcal{T}_z = \{t_f, 0\}. \quad (11)$$

For such scalar systems, the definitions 1 – 3 are reformulated.

Definition 4 *A function* $u = u(t,z)$, $(t,z) \in \mathcal{S}_z \triangleq \{(t,z): t \in [0,t_f), z \in R^z\}$, *is called an admissible feedback strategy for the system (9), if the corresponding closed-loop system has the unique absolutely continuous solution* $z(t)$, $t \in [0,t_f)$, *for any admissible disturbance* $v(t)$ *and for any initial condition* $z(t_0) = z_0$, $(t_0, z_0) \in \mathcal{S}_z$. *It is also assumed that there exists*

$$z(t_f) \triangleq \lim_{t \to t_f - 0} z(t). \quad (12)$$

Definition 5 *An admissible strategy* $u(t,z)$ *is called robust transferring from a given initial position* $(t_0, z_0) \in \mathcal{S}_z$ *to* $\mathcal{T}_z$, *if for any admissible* $v(t)$: $z(t_f) \in \mathcal{T}_z$. *It is called robust transferring from* $\mathcal{M}_z \subseteq \mathcal{S}_z$ *to* $\mathcal{T}_z$, *if it is robust transferring from any point* $(t_0, z_0) \in \mathcal{M}_z$ *to* $\mathcal{T}_z$,

For a given admissible strategy, let the set $\mathcal{M}_z^{\max} \subseteq \mathcal{S}_z$ be the maximal set, from which it is robust transferring to $\mathcal{T}_z$.

Definition 6 *The set* $\mathcal{C}_z = \mathcal{C}_z(u(\cdot)) \subseteq \mathcal{M}_z^{\max}$ *is called the robust controllability set of* $u(\cdot)$, *if for any initial point* $(t_0, z_0) \in \mathcal{C}_z$ *and any admissible disturbance, the time realization of* $u(t,z)$ *along the trajectory* $z = z(t)$ *satisfies the control constraint (2):*

$$|u(t, z(t))| \le \rho_u, \quad t \in [t_0, t_f). \quad (13)$$

Remark 1 Let the strategy $u(t,z)$ be robust transferring for the system (9) from $\mathcal{M}_z \subseteq \mathcal{S}_z$ to $\mathcal{T}_z$. In (Glizer and Turetsky, 2012), it is proved that if the strategy

$$\tilde{u}(t,x) = u(t, z(t,x)), \quad (14)$$

where $z(t,x)$ is given by (8), is admissible for the system (1), then it is robust transferring for this system from

$$\mathcal{M}_x = \{(t_0, x_0) : (t_0, z(t_0, x_0)) \in \mathcal{M}_z\} \quad (15)$$

to the target set (5). This yields that if $\mathcal{C}_z(u(\cdot))$ is the robust controllability set of $u(t,z)$, then the set

$$\mathcal{C}_x = \{(t_0, x_0) : (t_0, z(t_0, x_0)) \in \mathcal{C}_z(u(\cdot))\}, \quad (16)$$

is the robust controllability set of $\tilde{u}(t,x)$, given by (14).

Remark 1 allows to confine the following analysis only to the scalar case.

2.3. Previous Results

In this section, the main results of the book (Glizer and Turetsky, 2012, Chapters 2 – 3) on the construction of the robust controllability sets for linear and saturated linear robust transferring strategies are briefly summarized.

2.3.1. Linear Strategy

Let us introduce the characteristic numbers $N_1 \ge 0$, C_1, $N_2 \ge 0$, C_2 of the coefficient functions $h_1(t)$ and $h_2(t)$ of (9): assuming that that the limit exists,

$$\lim_{t \to t_f - 0} \frac{h_i(t)}{(t_f - t)^{N_i}} \triangleq C_i \ne 0, i = 1, 2. \quad (17)$$

It is assumed that

$$N_2 \ge N_1. \quad (18)$$

Consider the linear strategy

$$u(t,z) = K(t)z, \quad (19)$$

where the gain function $K(t)$ satisfies the following conditions:

(I) $K(t) \ne 0$ for $t \in [0, t_f)$.

(II) $K(t)$ is continuously differentiable for $t \in [0, t_f)$.

(III) one of two following limit conditions is satisfied:

$$\lim_{t \to t_f - 0} K(t) = +\infty, \quad (20)$$

or

$$\lim_{t \to t_f - 0} K(t) = -\infty, \quad (21)$$

(IV) there exists $N_K > 1$ such that

$$\lim_{t \to t_f - 0} \dot{K}(t)(t_f - t)^{N_K} = C \ne 0, \quad (22)$$

(V) either

$$N_K > N_1 + 2, \text{ and } CC_1 < 0, \quad (23)$$

or

$$N_K = N_1 + 2, \text{ and } CC_1 < -(N_K - 1)^2, \tag{24}$$

The set of all gains $K(t)$, satisfying the conditions (I) – (V), is denoted as $\mathcal{K}$. If the condition (18) is satisfied, then the linear strategy (19) with the gain $K(t) \in \mathcal{K}$ is robust transferring from $\mathcal{S}_z$ to $(t_f, 0)$.

Let us introduce the function

$$\mathcal{P}(t, K(t)) = \\ \dot{Z}^* - \rho_u(\text{sign}K(t))h_1(t) - \rho_v \,|\, h_2(t) \,|, \tag{25}$$

where

$$Z^*(t) \triangleq \frac{\rho_u}{|\, K(t) \,|}. \tag{26}$$

Assume that the set of zeros of the function $\mathcal{P}(t, K(t))$ on $(0, t_f)$ is finite (including empty). Due to this assumption, there exists $0 < \delta \le t_f$ such that two cases can be distinguished:

Case 1:

$$\mathcal{P}(t, K(t)) > 0, \; t \in (t_f - \delta, t_f). \tag{27}$$

Case 2:

$$\mathcal{P}(t, K(t)) < 0, \; t \in (t_f - \delta, t_f). \tag{28}$$

Let for a given $K(t)$, the set $\mathcal{T}$ consist of all distinct zeros of $\mathcal{P}(t, K(t))$ with positive slope. If $\mathcal{T} \ne \varnothing$, it can be written as 2, where $t_1 < t_2 < \ldots < t_p$. Let in this case $Z_i(t_0)$, $i = 1, \ldots, p$, be the solution of the terminal value problem

$$dZ \,/\, dt_0 = K(t_0)h_1(t_0)Z + \rho_v \,|\, h_2(t_0) \,|, \tag{29}$$

$$Z(t_i) = Z^*(t_i), \tag{30}$$

on the interval $[0, t_i]$. In Case 2, an additional function $Z_{p+1}(t_0)$ is defined as the solution of the equation (29) with the initial condition

$$Z_{p+1}(0) = \lim_{t \to t_f - 0} F(t), \tag{31}$$

where

$$F(t) \triangleq \frac{\rho_u - \rho_v \,|\, K(t) \,|\, \int_{t_0}^{t} G(t, \xi) \,|\, h_2(\xi) \,|\, d\xi}{|\, K(t) \,|\, G(t, 0)}, \tag{32}$$

$$G(t, \xi) = \exp\left(\int_{\xi}^{t} K(\eta)h_1(\eta)d\eta\right). \tag{33}$$

Let r be the maximal index ($r \in \overline{1, p}$ in Case 1 or $r \in \overline{1, p+1}$ in Case 2), such that the trajectory $z_0 = Z_r(t_0)$ intersects the t_0-axis, and $t_c = t_c(K(\cdot)) \in (0, t_r)$ be the last time moment such that $Z_r(t_c + 0) > 0$, $Z_r(t_c - 0) < 0$. If no trajectory intersects the t_0-axis, then $t_c = 0$. If $t_c = t_f$, then the robust controllability set $\mathcal{C}_z(K(\cdot))$ of the strategy (19) is empty.

Theorem 1 *Let for a given gain* $K(t) \in \mathcal{K}$, *Case 1 be valid. Then* $\mathcal{C}_z(K(\cdot)) \ne \varnothing$.

If $\mathcal{T} \ne \varnothing$, *then*

$$\mathcal{C}_z(K(\cdot)) = \{(t_0, z_0): \; t_c \le t_0 < t_f, \\ |\, z_0 \,|\le \min\{Z^*(t_0), Z_r(t_0), \ldots, Z_p(t_0)\}\}. \tag{34a}$$

If $\mathcal{T} = \varnothing$, *then*

$$\mathcal{C}_z(K(\cdot)) = \{(t_0, z_0): \; t_c \le t_0 < t_f, \; |\, z_0 \,|\le Z^*(t_0)\}. \tag{34b}$$

Theorem 2 *Let for a given gain* $K(t) \in \mathcal{K}$, *Case 2 be valid. If* $\mathcal{C}_z(K(\cdot)) \ne \varnothing$, *it is given as follows.*

If $\mathcal{T} \ne \varnothing$, *then*

$$\mathcal{C}_z(K(\cdot)) = \{(t_0, z_0): \; t_c \le t_0 < t_f, \\ |\, z_0 \,|\le \min\{Z^*(t_0), Z_r(t_0), \ldots, Z_p(t_0), Z_{p+1}(t_0)\}\}. \tag{35a}$$

If $\mathcal{T} = \varnothing$, *then*

$$\mathcal{C}_z(K(\cdot)) = \{(t_0, z_0): \; t_c \le t_0 < t_f, \\ |\, z_0 \,|\le \min\{Z^*(t_0), Z_{p+1}(t_0)\}\}. \tag{35b}$$

Remark 2 It follows from Theorems 1 and 2 that the robust controllability set $\mathcal{C}_z(K(\cdot))$ is symmetric with respect to the t_0-axis, and is described as

$$\mathcal{C}_z(K(\cdot)) = \\ \{(t_0, z_0): \; t_c \le t_0 < t_f, \; |\, z_0 \,|\le Z_b(t_0)\}. \tag{36}$$

The function $Z_b(t_0)$ denotes the respective boundary functions in (34) – (35).

2.3.2. Saturated Linear Strategy

Let us consider the saturation of (19) for $(t, z) \in \mathcal{S}_z$:

$$u_{sat}(t,z) = \mathrm{Sat}(K(t)z) = \begin{cases} \rho_u, & K(t)z > \rho_u, \\ K(t)z, & |K(t)z| \le \rho_u, \\ -\rho_u, & K(t)z < -\rho_u. \end{cases} \quad (37)$$

Remark 3 This saturated linear strategy respects by definition the control constraint along any trajectory of (9). Therefore, its robust controllability set $\mathcal{C}_z^{sat} = \mathcal{C}_z^{sat}(K(\cdot))$ consists of all the points $(t_0, z_0) \in \mathcal{S}_z$, from which this strategy is robust transferring. In general, the strategy (37) is not robust transferring from the entire set $\mathcal{S}_z$ to the target point $(t_f, 0)$. However, it is robust transferring at least from the robust controllability set $\mathcal{C}_z$ of the strategy (19). In this subset, $K(t)z$ is robust transferring and $|K(t)z(t)| \le \rho_u$. Thus, for any gain $K(t) \in \mathcal{K}$,

$$\mathcal{C}_z^{sat}(K(\cdot)) \supseteq \mathcal{C}_z(K(\cdot)). \quad (38)$$

The inclusion (38) is illustrated by Fig. 1, where $t_f = 4$. The strip $\mathcal{S}_z = \{(t_0, z_0) : t_0 \in [0,4],\ z_0 \in (-\infty, +\infty)\}$ is the robust transferrable set of a linear robust transferring strategy $u(t,z) = K(t)z$. The set, denoted as *I*, consists of all initial positions, from which the time realizations of the strategy $u = K(t)z(t)$ satisfy the constraint (2). This also implies that the saturated linear stratefgy $u = \mathrm{Sat}\left(K(t)z(t)\right)$ is robust transferring from this set. The set *II* is the set of all initial positions, for which $|K(t)z(t)| > \rho_u$, while the saturated linear strategy still remains robust transferring. The set *III* contains all the points of $\mathcal{S}_z$, for which $|K(t)z(t)| > \rho_u$ and the saturated linear strategy is not robust transferring. In other words, $I = \mathcal{C}_z(K(\cdot))$, $II = \mathcal{C}_z^{sat}(K(\cdot)) \setminus \mathcal{C}_z(K(\cdot))$ and $III = \mathcal{S}_z \setminus \mathcal{C}_z^{sat}(K(\cdot))$. From the practical viewpoint, the inclusion (38) means that implementing the saturated linear strategy is preferable than the corresponding linear strategy.

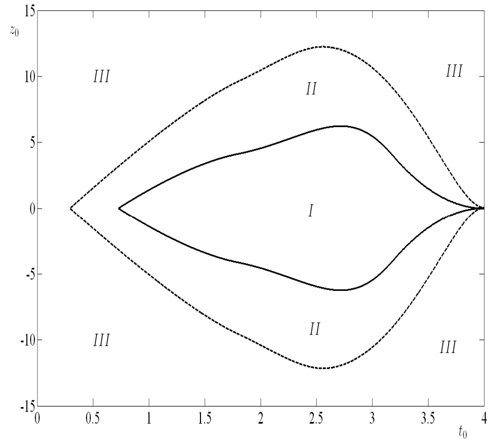

Figure 1: Illustration of the Inclusion (38)

Let $z(t; \bar{t}, \bar{z})$, $t \in [\bar{t}, t_f)$, denote the solution of the initial value problem

$$\dot{z} = h_1(t)u_{sat}(t,z) + h_2(t)v,\ \ z(\bar{t}) = \bar{z}. \quad (39)$$

Theorem 3 *If* $\mathcal{C}_z^{sat}(K(\cdot)) \ne \varnothing$, *it is given by*

$$\mathcal{C}_z^{sat}(K(\cdot)) = \{(t_0, z_0):\ t_0 \in [t_c^s, t_f), |z_0| \le Z_0^s(t_0)\}, \quad (40)$$

where

$$t_c^s = \min\{t_0 \in [0, t_f) : \exists\, z_0 \ge 0 : z(t_f; t_0, z_0) = 0\}, \quad (41)$$

$$Z_0^s(t_0) = z(t_0; t_c^s, z_0^s), \quad (42)$$

$$z_0^s = \max\{z_0 \ge 0 : z(t_f; t_c^s, z_0) = 0\}. \quad (43)$$

Introduce the function

$$Z_m(t) \triangleq \int_t^{t_f} (\rho_u |h_1(\xi)| - \rho_v |h_2(\xi)|) d\xi. \quad (44)$$

Theorem 4 *Let*

$$h_1(t) \ne 0,\ \ t \in [0, t_f). \quad (45)$$

and

$$Z_m(t) > 0,\ \ t \in [0, t_f). \quad (46)$$

Then

$$\mathcal{C}_z^{sat}(K(\cdot)) \subseteq \mathcal{C}_z^m \triangleq \{(t_0, z_0) : t_0 \in [0, t_f), |z_0| \le Z_m(t_0)\}. \quad (47)$$

Moreover,

$$\mathcal{C}_z^{sat}(K(\cdot)) = \mathcal{C}_z^m, \tag{48}$$

if and only if

$$\rho_u / |K(t)| \le Z_m(t), t \in [0, t_f). \tag{49}$$

Remark 4 Due to (Glizer and Turetsky, 2008), for any admissible robust transferring strategy $u(t,z)$:

$$\mathcal{C}_z(u(\cdot)) \subseteq \mathcal{C}_z^m. \tag{50}$$

This means that, subject to the conditions (45), (46) and (49), the robust controllability set $\mathcal{C}_z^{sat}(K(\cdot))$ is maximal for the system (9).

3. NEW RESULTS

For any $K(t) \in \mathcal{K}$, let define the sign-varying gain

$$\tilde{K}(t) = \begin{cases} -\mathrm{sign} h_1(t) |K(t)|, & h_1(t) \neq 0, \\ -\mathrm{sign} h_1(t+0) |K(t+0)|, & h_1(t) = 0, \end{cases}$$
$$t \in [0, t_f) \tag{51}$$

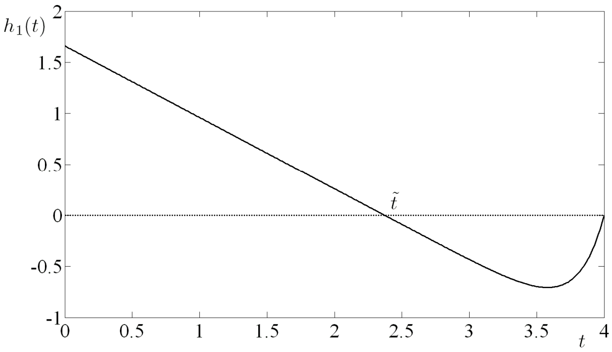

Figure 2a: Illustration of the Equation (51)

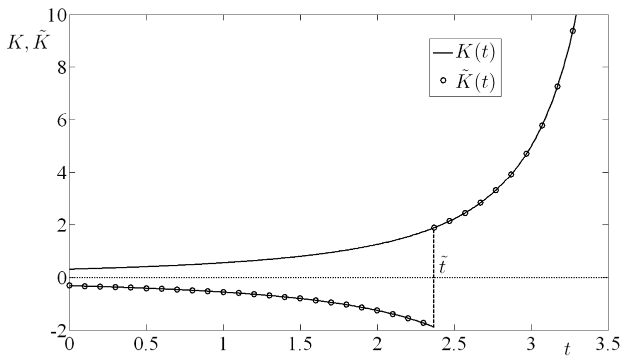

Figure 2b: Illustration of the Equation (51)

The construction of the gain $\tilde{K}(t)$ is illustrated by Fig. 2. The set of all such functions is denoted $\tilde{\mathcal{K}}$.

Remark 5 Due to (51), for all $\tilde{K}(t) \in \tilde{\mathcal{K}}$: $|\tilde{K}(t)| \in \mathcal{K}$.

Let extend the set of admissible gains to the set

$$\mathcal{K}_1 = \mathcal{K} \cup \tilde{\mathcal{K}}. \tag{52}$$

For $K(t) \in \mathcal{K}$, the robust controllability set of a linear and a saturated linear strategies is constructed by Theorems 1 – 2 and Theorem 3, respectively. In this section, the sets $\mathcal{C}_z(K(\cdot))$ and $\mathcal{C}_z^{sat}(K(\cdot))$ are constructed for $\tilde{K}(t) \in \tilde{\mathcal{K}}$.

3.1. Linear Strategy

Lemma 1 *Let* $K(t) \in \mathcal{K}$ *and* $\tilde{K}(t) \in \tilde{\mathcal{K}}$ *correspond to* $K(t)$ *by (51). Then for any initial position* $(t_0, z_0) \in \mathcal{S}_z$ *and for any admissible disturbance* $v(t)$, *the initial value problem*

$$\dot{z} = h_1(t)\tilde{K}(t)z + h_2(t)v, \; z(t_0) = z_0, \tag{53}$$

is equivalent to the initial value problem

$$\dot{z} = -|h_1(t)||K(t)|z + h_2(t)v, \; z(t_0) = z_0. \tag{54}$$

Proof. This lemma directly follows from the definition (51) of the gain $\tilde{K}(t)$.

□

Theorem 5 *Let* $K(t) \in \mathcal{K}$ *and* $\tilde{K}(t) \in \tilde{\mathcal{K}}$ *correspond to* $K(t)$ *by (51). Let* $h_1(t)$ has only a finite number of distinct zeros on the interval $[0, t_f]$. *Then the robust controllability set* $\mathcal{C}_z(\tilde{K}(\cdot))$ *is constructed by applying Theorems 1 –2 to the system*

$$\dot{z} = -|h_1(t)|u + h_2(t)v, \tag{55}$$

with the strategy

$$u(t,z) = |K(t)|z. \tag{56}$$

Moreover,

$$\mathcal{C}_z(\tilde{K}(\cdot)) \supseteq \mathcal{C}_z(K(\cdot)). \tag{57}$$

Proof. The first statement of the theorem is a direct consequence of Remark 5 and Lemma 1. In order to prove the inclusion (57), it is sufficient to show that if $(t_0, z_0) \in \mathcal{C}_z(K(\cdot))$, then $(t_0, z_0) \in \mathcal{C}_z(\tilde{K}(\cdot))$. For this, let us show that any trajectory of the system (53), starting from $(t_0, z_0) \in \mathcal{C}_z(K(\cdot))$, does not leave $\mathcal{C}_z(K(\cdot))$. Assume the opposite, i.e. that for some initial point $(t_0, z_0) \in \mathcal{C}_z(K(\cdot))$ and for some admissible disturbance $v(t)$, the trajectory $z_2(t; t_0, z_0, v(\cdot))$ of the system (53) leaves $\mathcal{C}_z(K(\cdot))$ through its upper boundary. This means that there exist the time moment $t_1 \in [t_0, t_f)$ and a number $\delta > 0$ such that $z_2(t_1; t_0, z_0, v(\cdot)) = Z_b(t_1)$ and for $t \in (t_1, t_1 + \delta)$:

$$z_2(t; t_0, z_0, v(\cdot)) > Z_b(t). \tag{58}$$

Note that

$$-|h_1(t)K(t)|z \le h_1(t)K(t)z, \quad z \ge 0. \tag{59}$$

Let $z_1(t; t_1, Z_b(t_1), v(\cdot))$ denote the trajectory of the system (9) for $u = K(t)z$ and the same disturbance $v(t)$ as in (53), starting from the boundary point $(t_1, Z_b(t_1))$. Then, by applying Differential Inequality Theorem (Hartman, 1964) to the systems (53) and (9) with $u = K(t)z$, and by using the inequality (59),

$$\begin{gathered} z_2(t; t_0, z_0, v(\cdot)) \le z_1(t; t_1, Z_b(t_1), v(\cdot)), \\ t \in (t_1, t_1 + \delta_1), \end{gathered} \tag{60}$$

where $0 < \delta_1 \le \delta$. By definition of the robust controllability set $\mathcal{C}_z(K(\cdot))$,

$$\begin{gathered} z_1(t; t_1, Z_b(t_1), v(\cdot)) < Z_b(t), \\ t \in (t_1, t_1 + \delta_1). \end{gathered} \tag{61}$$

The inequalities (60) – (61) contradict the inequality (58), meaning that the trajectory $z_2(t; t_0, z_0, v(\cdot))$ cannot leave $\mathcal{C}_z(K(\cdot))$ through its upper boundary.

The fact that it also cannot leave $\mathcal{C}_z(K(\cdot))$ through its lower boundary, is proved similarly by using the inequality

$$-|h_1(t)K(t)|z \ge h_1(t)K(t)z, \quad z < 0. \tag{62}$$

□

3.2. Saturated Linear Strategy

Similarly to Lemma 1 and Theorem 5 in the case of a linear strategy, the following lemma and theorem hold.

Lemma 2 *Let $K(t) \in \mathcal{K}$ and $\tilde{K}(t) \in \tilde{\mathcal{K}}$ correspond to $K(t)$ by (51). Then for any initial position $(t_0, z_0) \in \mathcal{S}_z$ and for any admissible disturbance $v(t)$, the initial value problem*

$$\dot{z} = h_1(t)\mathrm{Sat}(\tilde{K}(t))z + h_2(t)v, \;\; z(t_0) = z_0, \tag{63}$$

is equivalent to the initial value problem

$$\begin{gathered} \dot{z} = -|h_1(t)|Sat(|K(t)|z) + h_2(t)v, \\ z(t_0) = z_0. \end{gathered} \tag{64}$$

Theorem 6 *Let $K(t) \in \mathcal{K}$ and $\tilde{K}(t) \in \tilde{\mathcal{K}}$ correspond to $K(t)$ by (51). Let* $h_1(t)$ has only a finite number of distinct zeros on the interval $[0, t_f]$. *Then the robust controllability set $\mathcal{C}_z^{sat}(\tilde{K}(\cdot))$ is constructed by applying Theorem 3 to the system*

$$\dot{z} = -|h_1(t)|u + h_2(t)v, \tag{65}$$

with the strategy

$$u(t,z) = \mathrm{Sat}(|K(t)|z). \tag{66}$$

Moreover,

$$\mathcal{C}_z^{sat}(\tilde{K}(\cdot)) \supseteq \mathcal{C}_z^{sat}(K(\cdot)). \tag{67}$$

The following theorem is a direct consequence of Theorem 4 and Lemma 2.

Theorem 7 *Subject to the condition (46), the robust controllability set $\mathcal{C}_z^{sat}(\tilde{K}(\cdot))$ is maximal for the system (9) if and only if the inequality (49) is satisfied.*

4. INTERCEPTION PROBLEM

In this section, the results of Section 3 are applied to an interception problem with non-minimum phase controllers. Consider a planar engagement between two point-mass objects (pursuer and evader). The velocities V_p and V_e and the bounds of the lateral acceleration commands $a_p^{\max}$ and $a_e^{\max}$ of the objects are constant. The geometry of such planar engagement is presented in Fig. 3.

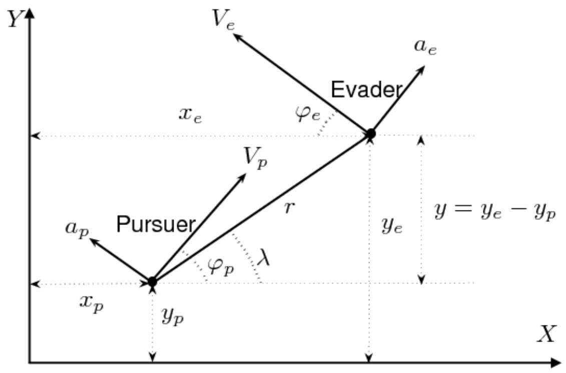

Figure 3: Interception Geometry

In this paper, in contrast with (Glizer and Turetsky, 2012; Shinar, 1981), it is assumed that the controllers of the pursuer and the evader are described by non-minimum phase transfer functions

$$H_i(s) = \frac{s - b_i}{1 + \tau_i s}, i = p, e, \tag{68}$$

where b_p, τ_p, b_e and τ_e are positive constants.

Assuming that the aspect angles φ_p and φ_e are small, the engagement can be modeled by the system (1). In this system, x_1 is the relative separation between the objects, normal to the initial line-of-sight; x_2 is the relative normal velocity. Due to the non-minimum phase form of the transfer functions (68), the variables x_3 and x_4 are connected to the lateral accelerations of the evader and the pursuer by

$$a_e = x_3 + \frac{1}{\tau_e} v, \tag{69}$$

$$a_p = x_4 + \frac{1}{\tau_p} u, \tag{70}$$

The controls of the pursuer u and the evader v are the lateral acceleration commands, satisfying the constraints (2) – (3) with $\rho_u = a_p^{\max}$ and $\rho_v = a_e^{\max}$, respectively. The final time is $t_f = r_0 / (V_p + V_e)$, where r_0 is the initial range between the objects. In this example, $V_p = 700$ m/s, $V_e = 800$ m/s, $r_0 = 6$ km, and, consequently, $t_f = 4$ s. The matrix A is

$$A(t) \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1/\tau_e & 0 \\ 0 & 0 & 0 & -1/\tau_p \end{bmatrix}, \tag{71}$$

the vectors b and c are

$$\begin{aligned} b(t) &\equiv (0, -1/\tau_p, 0, -(1 + b_p/\tau_p))^T, \\ c(t) &\equiv (0, 1/\tau_e, -(1 + b_e/\tau_e), 0)^T, \end{aligned} \tag{72}$$

$$\begin{aligned} f(t) &\equiv 0, \ x_0 = (0, x_{20}, 0, 0)^T, \\ x_{20} &= V_e \varphi_e(0) - V_p \varphi_p(0). \end{aligned} \tag{73}$$

The objective of the pursuer is to nullify the miss distance $|x_1(t_f)|$, i.e. in the target hyperplane, $d = (1,0,0,0)^T$, $d_0 = 0$.
In the scalar system (9):

$$h_1(t) = (1 + \tau_p b_p)\psi((t_f - t)/\tau_p) - \frac{t_f - t}{\tau_p}, \tag{74}$$

$$h_2(t) = (1 + \tau_e b_e)\psi((t_f - t)/\tau_e) - \frac{t_f - t}{\tau_e}, \tag{75}$$

where $\psi(\xi) \triangleq \exp(-\xi) + \xi - 1$.

For these coefficient functions, $N_1 = N_2 = 1$, $C_1 = -1/\tau_p$, $C_2 = -1/\tau_e$, i.e. the condition (18) holds.

Proposition 1 *If*

$$b_p > \frac{1 - \exp(-t_f/\tau_p)}{\tau_p(\exp(-t_f/\tau_p) + t_f/\tau_p - 1)}, \tag{76}$$

then the function $h_1(t)$, given by (74), changes its sign once in the interval $(0, t_f)$. Moreover, if the inequality(76) is not satisfied, then $h_1(t)$ is sign-constant for $t \in (0, t_f)$.

Proof. Let start with the first statement of the theorem. The inequality (76) is equivalent to the inequality

$$h_1(0) > 0. \tag{77}$$

Note that

$$h_1(t_f) = 0. \tag{78}$$

The derivative of the function $h_1(t)$ is

$$\dot{h}_1(t) = \left(\frac{1}{\tau_p} + b_p\right)\exp(-(t_f - t)/\tau_p) - b_p, \tag{79}$$

yielding

$$\dot{h}_1(t_f) = \frac{1}{\tau_p} > 0. \quad (80)$$

The relations (77) – (78) and (80) guarantee that the function $h_1(t)$ changes its sign at least once in the interval $(0, t_f)$.

Let prove that $h_1(t)$ changes its sign exactly once in the interval $(0, t_f)$. Taking into account the equation (85), it is sufficient to show that the derivative $\dot{h}_1(t)$ has no more than one zero in the interval $(0, t_f)$. Indeed, due to (79), the equation $\dot{h}_1(t) = 0$ can be rewritten as

$$\exp(-(t_f - t)/\tau_p) = \tau_p b_p / (1 + \tau_p b_p), \quad (81)$$

which has no more than one zero in the interval $(0, t_f)$. This completes the proof of the first statement of the proposition. The second statement is proved by similar arguments. □

Consider the linear feedback strategy (19) with the gain

$$K(t) = A / (t_f - t)^2, \quad (82)$$

where $A > 0$. Note that the gain (82) satisfies the conditions (I) – (III). The characteristic numbers of this gain, defined by (22), are $N_K = 3$, $C = 2A$. Note that $N_K = N_1 + 2$ and the condition $CC_1 = -2A/\tau_p < -(N_K - 1)^2 = -4$ is satisfied for

$$A > 2\tau_p. \quad (83)$$

Thus, for such a gain the conditions (IV) – (V) are also satisfied and $K(t) \in \mathcal{K}$. Therefore, the strategy

$$u(t, z) = Az / (t_f - t)^2, \quad (84)$$

where A satisfies (83), is robust transferring from $\mathcal{S}_z = \{(t, z) : t \in [0, t_f),\ z \in R^1\}$ to $(t_f, 0)$.

Example. Let $t_f = 4$ s, $\tau_p = 0.2$ s, $a_p^{\max} = 30$ m/s^2, $b_e = 0.5$ s^{-1}, $\tau_e = 0.2$ s, $a_e^{\max} = 10$ m/s^2, $A = 5$ s. For these parameters, the inequality (76) becomes $b_p > 0.263$. In this example, b_p is chosen as $b_p = 0.7$ s^{-1}. The closed-loop system, corresponding to the gain (82), is

$$\dot{z} = \frac{5h_1(t)}{(4-t)^2} z + h_2(t)v. \quad (85)$$

The graph of the function $h_1(t)$ is depicted in Fig. 2. It is seen that this function changes the sign from positive to negative at $t = \tilde{t} = 2.37$. Thus, due to (51),

$$\tilde{K}(t) = \begin{cases} -5/(4-t)^2, & 0 \le t < \tilde{t}, \\ 5/(4-t)^2, & \tilde{t} \le t < 4. \end{cases} \quad (86)$$

The equivalent closed-loop system (54), corresponding to the gain (86), is

$$\dot{z} = -5|h_1(t)|z/(4-t)^2 + h_2(t)v. \quad (87)$$

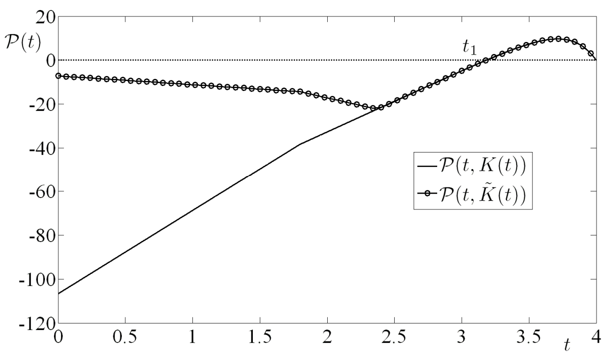

Figure 5: Interception Problem: the Functions $\mathcal{P}(t, K(t))$ and $\mathcal{P}(t, \tilde{K}(t))$

In Fig. 5, the functions $\mathcal{P}(t, K(t))$ and $\mathcal{P}(t, \tilde{K}(t))$ are depicted. It is seen that Case 1 (see (27)) is valid for both gains and $\mathcal{T} = \{t_1\} = \{3.19\}$. In Fig. 6, the robust controllability sets of the linear control strategies $\mathcal{C}_z(K(\cdot))$ and $\mathcal{C}_z(\tilde{K}(\cdot))$ are depicted, demonstrating the advantage of the discontinous-sign gain $\mathcal{C}_z(\tilde{K}(\cdot)) \supset \mathcal{C}_z(K(\cdot))$.

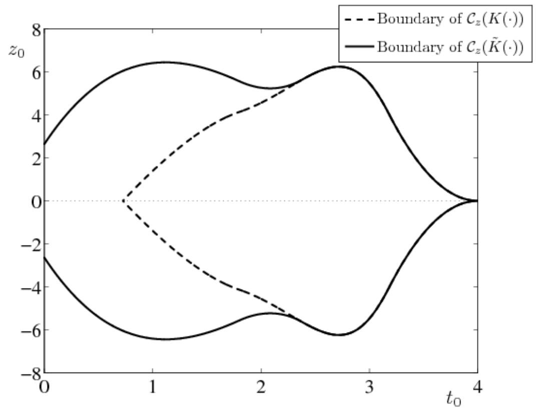

Figure 6: Robust Controllability Sets of the Linear Control Strategies $\mathcal{C}_z(K(\cdot))$ and $\mathcal{C}_z(\tilde{K}(\cdot))$

In Fig. 7, the robust controllability sets of the saturated linear control strategies $\mathcal{C}_z^{sat}(K(\cdot))$ and $\mathcal{C}_z^{sat}(\tilde{K}(\cdot))$ are depicted, showing that, similarly to the case of linear strategies, $\mathcal{C}_z^{sat}(\tilde{K}(\cdot)) \supset \mathcal{C}_z^{sat}(K(\cdot))$.

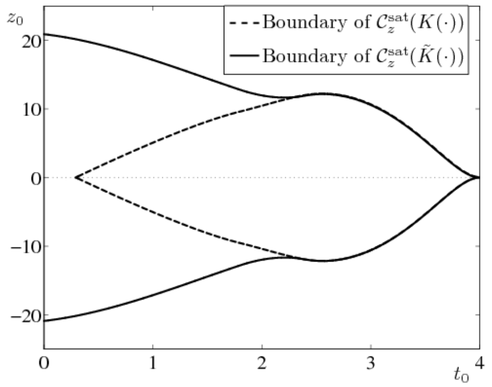

Figure 7: Robust Controllability Sets of the Saturated Linear Control Strategies $\mathcal{C}_z^{sat}(K(\cdot))$ and $\mathcal{C}_z^{sat}(\tilde{K}(\cdot))$

5 CONCLUSIONS

In the paper, a linear controlled system having a sign-varying control coefficient with bounded disturbance is considered. Using for such systems a linear, or a saturated linear control strategy with continous-sign gain leads to a small (sometimes even empty) robust controllability set. Earlier results for constructing the robust controllability sets of linear and saturated linear transferring strategies are extended to the case of a sign-varying discontinuous gain. Such an extension is based on the reducing the sytem with discontinuous gain control to the equivalent system with a corresponding continuous gain control. It is shown that by replacing the continous gain with a properly chosen discontinous gain the robust controllability set is substantially enlarged. It is also shown that the robust controllability set of a saturated linear control strategy is larget than the robust controllability set of the corresponding linear control strategy. These results are illustrated by the example of an interception problem with non-minimum phase controllers of the pursuer and the evader.

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