MODEL COMPLEXITY OF A CANTILEVER BEAM: AN ENERGY-BASED APPROACH

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ABSTRACT

The cantilever beam is a component widely used in numerous engineering systems with its geometric and material properties varying depending on the application. Calculating the dynamic behavior of a cantilever beam is a challenging task since the critical physical phenomena and interactions vary significantly based on the geometry of the beam. There exist a number of theories/models that can be used to predict the transverse motion of a cantilever beam of which the two most commonly used are the Timoshenko and Euler-Bernoulli theories. The Euler-Bernoulli theory is simpler and thus preferred, however, depending on the beam's parameters and operating conditions this model can lead to erroneous results and thus the more complex Timoshenko theory must be used. Currently, selecting the theory to use depends on heuristics or rules that are based on experience and the accuracy requirements of the predictions. It is the purpose of this paper to address the model complexity of a cantilever beam through a systematic modeling methodology.

The paper presents a new approach for selecting the appropriate theory to use in modeling a cantilever beam. The beam is discretized through the finite segment approach and modeled using the bond graph formulation. The previously developed activity metric is then used to determine which of the inertial and stiffness effects, of the more complex Timoshenko theory, need to be included in the model in order to have accurate predictions of its dynamic behavior. An illustrative example is provided to demonstrate the new methodology.

Keywords: Cantilever beam, Timoshenko beam theory, model reduction, activity metric.

1. INTRODUCTION

Modeling and simulation have yet to achieve wide utilization as commonplace engineering tools. One reason for this is that current modeling and simulation techniques are inadequate. Specifically, a major disadvantage is that they require sophisticated users who are often not domain experts and thus lack the ability to effectively utilize the model and simulation tools to uncover the important design trade-offs. Another drawback is that models are often large and complicated with many parameters, making the physical interpretation of the model outputs, even by domain experts, difficult. This is particularly true when "unnecessary" features are included in the model.

A variety of algorithms have been developed and implemented to help automate the production of proper models of dynamic systems. Wilson and Stein (1995) developed MODA (Model Order Deduction Algorithm) that deduces the required system model complexity from subsystem models of variable complexity using a frequency-based metric. They also defined proper models as the models with physically meaningful states and parameters that are of necessary but sufficient complexity to meet the engineering and accuracy objectives. Additional work on deduction algorithms for generating proper models in an automated fashion, has been reported by Ferris et al. (1998), Ferris and Stein (1995) and Walker et al. (1996). The above algorithms have also been implemented in an automated modeling computer environment (Stein and Louca, 1996).

In an attempt to overcome the limitations of the frequency-based metrics, Louca et al. (1997) introduced a new model reduction technique that also generates proper models. This approach uses an energy-based metric (element activity) that in general, can be applied to nonlinear systems (Louca et al., 2010), and considers the importance of all energetic elements (generalized inductance, capacitance and resistance). The contribution of each energy element in the model is ranked according to the activity metric under specific excitation. Elements with small contributions are eliminated in order to produce a reduced model using a systematic methodology called Model Order Reduction Algorithm (MORA). The activity metric was also used as a basis for even further reduction, through partitioning the model into smaller and decoupled submodels (Rideout et al. 2007).

Such modeling approaches should be able to handle real mechanical systems that typically include distributed parameter (continuous) components, e.g. rods, beams, plates, etc. Frequently, modeling objectives and assumptions allow the lumping of continuous component properties into ideal energy elements that lead to a dynamic model described by a set of ordinary differential equations. However, when property lumping is not acceptable, modeling of a continuous component requires a different approach since its inertial, compliance and resistive properties are spatially distributed and cannot be lumped into single equivalent elements. The dynamic behavior of continuous components is thus described by partial differential equations with derivatives in both time and space. Another approach that is considered in this work is the modeling of a continuous component with finite segments that are spatially distributed. This is an approximation for which the accuracy is a function of the

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number of segments. The model accuracy improves as the number of segments increases. Model accuracy and the required number of segments can be addressed using a frequency-based metric (Ferris et al., 1998).

Beyond the physical-based modeling, modal decomposition is also used to model and analyze continuous and discrete systems (Meirovitch, 1967). One of the advantages of modal decomposition is the ability to straightforwardly adjust (i.e., reduce) model complexity since all modes are orthogonal to each other. The reduction of such modal decomposition models is mostly based on frequency, and the user defined frequency range of interest (FROI) determines the frequencies that are important for a specific scenario. In this case, modes with frequencies within the FROI are retained in the reduced model and modes outside this range are eliminated. As expected, mode truncation introduces error in the predictions that can be measured and adjusted based on the accuracy requirements (Li and Gunter, 1981; Liu et al., 2000).

The element activity is another metric that has more flexibility than frequency-based metrics, which address the issue of model complexity by only adding compliant elements, leaving unaccounted the importance of inertial and resistive elements. In contrast, the activity metric considers the importance of all energetic elements, and therefore, the significance of all energy elements in the model can be quantified. It is the purpose of this work to develop a new methodology using the activity metric for addressing the model complexity of distributed parameter systems and specifically cantilever beams. The methodology is specifically developed using the finite segment approximation and the goal is to identify the physical phenomena to be included in each segment in order to accurately predict the dynamic behavior.

This paper is organized as follows: first, background about the energy-based activity metric is provided, along with the reduction algorithm. Next, the equation formulation for a finite segment Timoshenko beam is presented along with the closed-form expressions of the steady state activities. Then the complexity of a cantilever beam is analyzed using MORA. Finally, in the last section, discussion and conclusions are given.

2. BACKGROUND

The original work on the energy-based metric for model reduction is briefly described here for convenience. More details, extensions, and applications of this approach are given in previous publications (Louca and Stein, 2002; Louca et al., 2004; Louca and Stein, 2009; Louca et al., 2010). The main idea behind this model reduction technique is to evaluate the "element activity" of individual energy elements in a full system model under a stereotypical set of inputs and initial conditions. The activity of each energy element establishes a hierarchy for all elements. Those below a user-defined threshold of acceptable level of activity are eliminated from the model. A reduced model is then generated and a new set of governing differential equations is derived.

The activity metric has been previously formulated for systems with nonlinearities in both the element constitutive laws and junction structure. In this work, the activity metric is applied to linear systems for which analytical expressions for the activity can be derived, and therefore, avoid the use of numerical time integration that could be cumbersome. The analysis is further simplified if, in addition to the linearity assumption, the system is assumed to have a single sinusoidal excitation, and only the steady state response is examined. These assumptions are motivated from Fourier analysis where an arbitrary function can be decomposed into a series of harmonics. Using this frequency decomposition, the activity analysis can be performed as a function of frequency in order to study the frequency dependency of element activity in a dynamic system.

2.1. Element Activity for Linear Systems

A measure of the power response of a dynamic system, which has physical meaning and a simple definition, is used to develop the modeling metric, element activity (or simply "activity"). Element activity, A, is defined for each energy element as:

$$A = \int_0^\tau \left| \mathcal{P}(t) \right| dt \tag{1}$$

where $\mathcal{P}(t)$ is the element power and τ is the time over which the model has to predict the system behavior. The activity has units of energy, representing the amount of energy that flows in and out of the element over the given time τ. The energy that flows in and out of an element is a measure of how active this element is (how much energy passes through it), and consequently the quantity in Eq. (1) is termed activity. Activity can be defined independent of the energy domain, type of energy element or nonlinearities.

The activity is calculated for each energy element based on the system response. In the case that the system is modeled using a bond graph formulation, the state equations are derived using the multi-port bond graph representation (Borutzky, 2004; Brown, 2006; Karnopp et al., 2006; Rosenberg and Karnopp, 1983). In addition, when a system has a single input and linear junction structure and constitutive laws, the state equations are linear time invariant and have the following general form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{2}$$

where, $\mathbf{A} \in \mathbb{R}^{m \times m}, \mathbf{b} \in \mathbb{R}^m$ are the state space matrices, $\mathbf{x} \in \mathbb{R}^m$ is the state vector, $u \in \mathbb{R}$ is the input, and m is the number of independent states.

For the above system appropriate outputs are defined in order to calculate the power of each energy element in the model using the constitutive law of each element. For convenience, the outputs are selected to be generalized flow, effort, and flow for inertial, compliant, and resistive elements, respectively. The dual effort or flow needed for calculating the power is derived from the output variables

and constitutive laws. The output vector for this set of variables has the form:

$$\mathbf{y} = \begin{Bmatrix} \mathbf{f}_I \\ \hdashline \mathbf{e}_C \\ \hdashline \mathbf{f}_R \end{Bmatrix} \tag{3}$$

where $\mathbf{y} \in \mathbb{R}^k$ and $\mathbf{f}_I \in \mathbb{R}^{k_I}, \mathbf{e}_C \in \mathbb{R}^{k_C}$, and $\mathbf{f}_R \in \mathbb{R}^{k_R}$. The variables k_I, k_C, and k_R represent the number of inertial, compliant, and resistive elements, respectively. The total number of energy elements is $k = k_I + k_C + k_R$. Using the output variables set in Eq. (3), the output equations can be written as:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{d}u \tag{4}$$

where $\mathbf{C} \in \mathbb{R}^{k \times m}, \mathbf{d} \in \mathbb{R}^k$ are the output state space matrices. Note that the output vector is defined such that the required variables of the inertial elements are first, followed by the variables of compliant and then resistive elements.

Given this set of output variables the missing efforts or flows, needed for calculating the element power, are computed from the linear constitutive laws of each type of energy element as shown below:

$$\begin{aligned} &\textbf{\textit{I}}: \quad p_I = r_I f_I \Leftrightarrow e_I = \dot{p}_I = r_I \dot{f}_I \\ &\textbf{\textit{C}}: \quad q_C = r_C e_C \Leftrightarrow f_C = \dot{q}_C = r_C \dot{e}_C \\ &\textbf{\textit{R}}: \quad e_R = r_R f_R \end{aligned} \tag{5}$$

where r_I, r_C, r_R are known constants representing the linear constitutive law coefficients of inductance, compliance and resistance, respectively. For more compact expressions a vector, $\mathbf{r} \in \mathbb{R}^k$, with all the linear constitutive law coefficients is introduced as shown below:

$$\mathbf{r} = \begin{Bmatrix} \mathbf{r}_I \\ \hdashline \mathbf{r}_C \\ \hdashline \mathbf{r}_R \end{Bmatrix} \tag{6}$$

where $\mathbf{r}_I \in \mathbb{R}^{k_I}$, $\mathbf{r}_C \in \mathbb{R}^{k_C}$, and $\mathbf{r}_R \in \mathbb{R}^{k_R}$.

Finally, the power needed for calculating the activity of each element, as defined in Eq. (1), is computed as the product of generalized effort and flow. By using Eq. (5) the following expressions for the power of each element type are derived:

$$\begin{aligned} &\textbf{\textit{I}}: \mathcal{P}_I = e_I f_I = r_I f_I \dot{f}_I \\ &\textbf{\textit{C}}: \mathcal{P}_C = e_C f_C = r_C e_C \dot{e}_C \\ &\textbf{\textit{R}}: \mathcal{P}_R = e_R f_R = r_R f_R f_R = r_R f_R^2 \end{aligned} \tag{7}$$

The expressions for element power in Eq. (7) are generalized with the use of the defined structure of the output vector in Eq. (3) and parameter vector in Eq. (6). Thus, the power for energy storage elements (inertial and compliant) is given by Eq. (8) and for energy dissipation elements (resistive) in Eq. (9).

$$\mathcal{P}_i = r_i y_i \dot{y}_i, \quad i = 1, \ldots, k_I + k_C \tag{8}$$

$$\mathcal{P}_i = r_i y_i^2, \quad i = k_I + k_C + 1, \ldots, k \tag{9}$$

The above element power is then used to calculate the element activity based on its definition in Eq. (1). Element parameters are assumed to be constant thus the activity for each element is given by:

$$\begin{aligned} A_i &= \int_0^\tau \left| \mathcal{P}_i \right| = r_i \int_0^\tau \left| y_i \dot{y}_i \right| dt, \quad i = 1, \ldots, k_I + k_C \\ A_i &= \int_0^\tau \left| \mathcal{P}_i \right| = r_i \int_0^\tau \left| y_i^2 \right| dt, \quad i = k_I + k_C + 1, \ldots, k \end{aligned} \tag{10}$$

2.2. Activity for Single Harmonic Excitation

The time response of the output vector, $\mathbf{y}(t)$, in Eq. (3) is required in order to complete the calculation of the element power. For nonlinear systems, numerical integration is typically used to calculate the system response; however, in this case linear system analysis can be used to obtain closed form expressions. In addition, for the purposes of this work, the excitation is assumed to be a single harmonic given by:

$$u(t) = U \sin(\omega t) \tag{11}$$

where $U \in \mathbb{R}$ is the amplitude of the excitation and ω is the excitation frequency. The steady state response of the linear system in Eq. (2) and (4), and for the excitation in Eq. (11), is calculated using linear system analysis theory. This gives the following closed form expression:

$$y_i(t, \omega) = U Y_i(\omega) \cdot \sin\left(\omega t + \varphi_i(\omega)\right), \; i = 1, \ldots, k \tag{12}$$

where $Y_i(\omega)$ and $\varphi_i(\omega)$ are the steady state amplitude and phase shift, respectively that can be easily calculated from the state space matrices using linear systems theory.

Within the context of this analysis, the output $y_i(t, \omega)$ in Eq. (12) is either an effort or a flow that is used to calculate the power of each element in Eq. (7). Finally, the activity can be calculated by Eq. (1), but first the upper bound of this integral must be specified. For this case, the steady state and periodicity of the response are exploited. A periodic function repeats itself every T seconds, and therefore, a single period of this function contains the required information about the response. Thus, the upper bound of the integral is set to one period of the excitation, $\tau = T = 2\pi/\omega$. Therefore, the steady state activity for the energy storage elements is given by:

$$\begin{aligned} A_i^{ss}(\omega) &= r_i \int_0^T \left| y_i \dot{y}_i \right| dt \\ &= \frac{1}{2} r_i U^2 Y_i^2(\omega) \omega \int_0^T \left| \sin\left(2\left(\omega t + \varphi_i(\omega)\right)\right) \right| dt \\ &\Rightarrow A_i^{ss}(\omega) = 2 r_i U^2 Y_i^2(\omega) \end{aligned} \tag{13}$$

and for energy dissipation elements by:

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$$A_i^{ss}(\omega) = r_i \int_0^T \left| y_i^2 \right| dt$$
$$= r_i U^2 Y_i^2(\omega) \int_0^T \left| \sin^2\left(\omega t + \varphi_i(\omega)\right) \right| dt \qquad (14)$$
$$\Rightarrow A_i^{ss}(\omega) = \frac{\pi r_i U^2 Y_i^2(\omega)}{\omega}$$

The above simple closed form expressions can be used to calculate the activity of energy elements for a given single harmonic excitation. These expressions are proportional to the square of the amplitude, however, they have no dependency on the phase shift that is eliminated through the integration. The superscript 'ss' in Eq. (13)-(14) denotes the activity under a steady state harmonic response. Note that the activity for both energy storage and energy dissipation elements is a function of the excitation frequency but not the phase shift.

2.3. Activity Index and MORA

The activity as defined in Eq. (1) is a measure of the absolute importance of an element as it represents the amount of energy that flows through the element over a given time period. In order to obtain a relative measure of the importance, the element activity is compared to a quantity that represents the "overall activity" of the system. This "overall activity" is defined as the sum of all the element activities of the system, is termed total activity (A^{Total}) and is given by:

$$A^{Total}(\omega) = \sum_{i=1}^{k} A_i(\omega) \qquad (15)$$

where A_i is the activity of the i^{th} element given by Eq. (1). Thus a normalized measure of element importance, called the element activity index or just activity index, is defined as:

$$AI_i^{ss}(\omega) = \frac{A_i(\omega)}{A^{Total}(\omega)} = \frac{A_i(\omega)}{\sum_{i=1}^{k} A_i(\omega)} \qquad (16)$$

The activity index, $AI_i^{ss}(\omega)$, is calculated for each element in the model and it represents the portion of the total system energy that flows through a specific element. The input amplitude, U, does not appear in any of the element activity indices since all element activities are proportional to the square of the amplitude.

With the activity index defined as a relative metric for addressing element importance, the Model Order Reduction Algorithm (MORA) is constructed. The first step of MORA is to calculate the activity index for each element in the system for a given system excitation and initial conditions. Next, the activity indices are sorted to identify the elements with high activity (most important) and low activity (least important). With the activity indices sorted, the model reduction proceeds given the desired engineering specifications. These specifications are defined by the modeler who then converts them into a threshold β of the total activity (e.g., 99%) that he or she wants to include in the reduced model. This threshold defines the borderline between the retained and eliminated model elements. The elimination process is shown in Figure 1 where the sorted activity indices are summed starting from the most important element until the specified threshold is reached. The element which, when included, increments the cumulative activity above the threshold, is the last element to be included in the reduced model. The elements that are above this threshold are removed from the model, e.g., when using the bond graph formulation delete the corresponding energy element and connected bond.

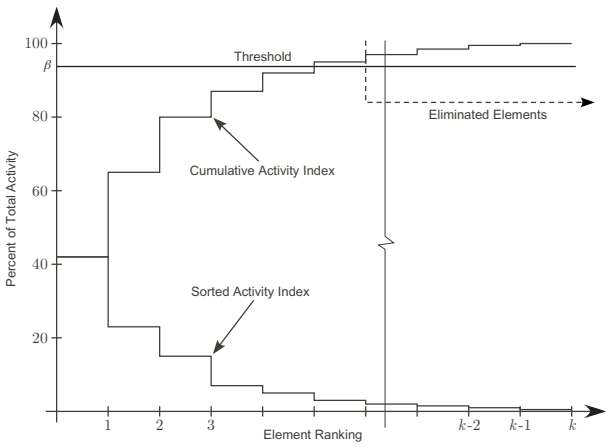

Figure 1: Activity index sorting and elimination.

3. TIMOSHENKO BEAM MODEL

The state space representation used in the previous section assumes that real components exhibit only inertial, compliant, or resistive behavior. This means that the dynamic behavior of a component can be lumped and modeled as a single inertial, compliant or resistive energy element. This can be a valid assumption for many components, however, real system components can possess all dynamic properties (inertial, compliant, resistive) simultaneously. In addition, these properties may vary or be distributed spatially. In these cases, a lumped parameter modeling approach cannot be used since it will produce erroneous predictions. These components must be considered as continuous and require a different modeling approach.

Models of continuous systems can be developed using solid mechanics techniques, which lead to Partial Differential Equations (PDE) with derivatives in both space and time (Bauchau and Craig, 2009; Genta, 2009; van Rensburg and van der Merwe, 2006; Li, 2008). The continuous cantilever beam used in this work is shown in Figure 2, where its transverse motion is considered when excited with a vertical load at its free end. The motion of a given gross section, $w(x,t)$ and $\varphi(x,t)$, from its undeformed state varies with time and location thus having PDEs describing its motion. Note that due to the rotation φ, a cross section does not remain normal to the neutral axis according to the Timoshenko beam theory that is used in this work. One method for solving these PDEs is separation of variables, which produces a modal

expansion solution (Meirovitch, 1967). This approach can also be combined with other lumped parameter elements in order to model a real system that consists of both lumped and distributed parameter components (Karnopp et al., 2006). An analysis of the advantages and disadvantages of this approach is beyond the scope of this work, however, it is safe to say that the solution of PDEs is more cumbersome than the solution of ordinary differential equations that describe the behavior of lumped parameters system.

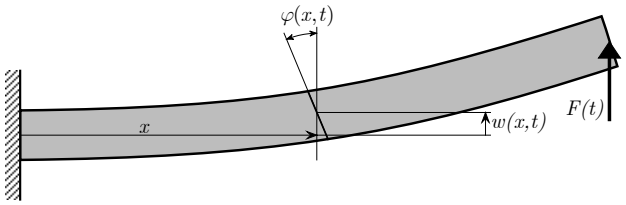

Figure 2: Cantilever beam transverse vibration.

A different approach for modeling the transverse vibration of a cantilever beam is to divide it into segments of equal length. This approach is motivated by the procedure for deriving the PDEs describing the motion of a beam. Each of these segments has linear inertial and compliant properties that can be determined from solid mechanics theory. Shear effects and rotational inertial effects are also considered, which results in a more generic model that is valid for a larger range of geometric parameters. This is known as the Timoshenko beam model, which is usually used for non-slender beams in order to get accurate model predictions. The use of this more complex model using the Timoshenko beam theory is also mandated from the use of MORA in the process of determining the appropriate model complexity. In this approach the most complex model is first developed, and then MORA is used to identify what is actually needed in order to reach a reduced model with accurate predictions.

The ideal physical model under these assumptions is shown in Figure 3 where the beam is divided into n segments. This model approaches the partial differential equations of the continuous system, as the number of segments approaches infinity. However, it is difficult to predict the number of segments required to achieve a given level of accuracy. It is well known that a large number of segments is required for accurately predicting low frequency dynamics. For the purposes of this work the number of segments is chosen based on previous research, such that the model accurately predicts low frequency dynamics that are considered in this work (Louca, 2014; Louca, 2015). With the given number of segments, the physical phenomena to be included in each segment, for the model to accurately predict the dynamic behavior, will then be identified using the proposed methodology in this paper.

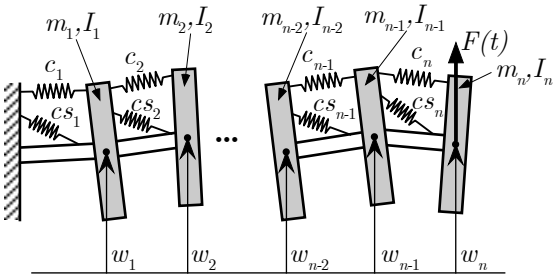

Figure 3: Ideal physical model of a Timoshenko beam.

For calculating the constitutive law parameters of the energy storage elements, the beam is assumed to have density ρ, Young's modulus E, shear modulus G, length L, cross sectional area A and cross sectional moment of inertia I. Given these physical parameters of the beam, the element parameters in the above linear model are given by the expressions below:

$$\begin{aligned} m_i &= \rho A \Delta x, \ i = 1,\ldots,n \\ I_i &= \rho I \Delta x, \\ c_i &= \frac{\Delta x}{EI} \\ cs_i &= \frac{\Delta x}{\kappa G A} \end{aligned} \tag{17}$$

where $\Delta x = L/n$ is the length of each segment, κ is a dimensionless constant that accounts for the non-uniform distribution of the shear stress and depends on the shape

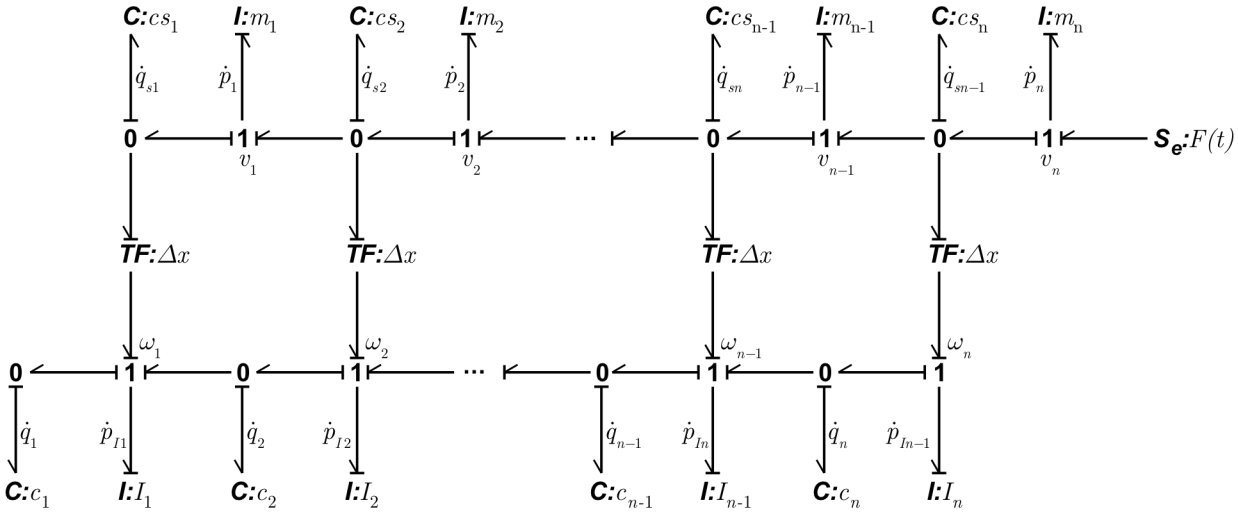

Figure 4: Bond graph model of a Timoshenko beam.

of the cross-section. The inertial parameters m_i and I_i represent the linear and rotational inertia of each segment, respectively. The parameters c_i and cs_i represent the bending and shear compliance between two segments, respectively. The beam is assumed to have no energy losses therefore there are no damping elements in the model. These parameters are used to define the parameter vector as defined in Eq. (6).

For developing the dynamic equations, the bond graph formulation is used. Bond graphs provide the power topography of the system and it is a natural selection for implementing the power-based activity metric. The bond graph model of the ideal physical model as shown in Figure 3 is developed and given in Figure 4. The bond graph has $4n$ independent state variables since each segment is modeled by 4 independent energy storage elements and its state vector has the form $\mathbf{x} = \{p_1,\ldots,p_n,p_{I1},\ldots,p_{In},q_1,\ldots,q_n,q_{s1},\ldots,q_{sn}\}^T$. The transverse velocity of the each mass, v_i, represents the velocity at a given location of the continuous beam and Eq. (18) expresses the relation between the discrete and continuous variables. The other kinematic variable of the model, ω_i, is the rotation at a given location and its relation to the continuous variable is given in Eq. (19).

$$v_i(t) = \dot{w}_i = \dot{w}\left(i\Delta x, t\right) \tag{18}$$

$$\omega_i(t) = \dot{\varphi}_i = \dot{\varphi}\left(i\Delta x, t\right) \tag{19}$$

In addition, for easy calculation of the output equations that required for calculating power, the state equations are derived using the multi-port approach (Rosenberg, 1971). According to this approach, the state space and input matrices are given by:

$$\mathbf{A} = \mathbf{J}_{SS}\,\mathbf{S}, \quad \mathbf{b} = \mathbf{J}_{SU} \tag{20}$$

The output matrices, as defined in Eq. (4), that are required for calculating the power flow into the energy elements are given by:

$$\mathbf{C} = \mathbf{S}, \quad \mathbf{d} = \mathbf{0} \tag{21}$$

The output vector according to the analysis in the previous section is given by $\mathbf{y} = \{f_1,\ldots,f_{2n},e_1,\ldots,e_{2n}\}^T$. The dimensions of the state space matrices as defined in the previous section are $m = 4n$ and $k = 4n$. Based on this set of state variables, the junction structure matrices, $\mathbf{J}_{SS}$ and $\mathbf{J}_{SU}$, are derived and given in the Appendix. The above equations and junction structure matrices are simplified since the model has no resistive elements.

For the above model with n segments the steady-state response is first calculated using Eq. (12) and based on the state space equations in Eq. (20)-(21). Then the element activity is calculated from Eq. (13) and (14), which gives the following expression for the energy storage elements of the model:

$$A_i^{ss}\left(\omega\right) = 2r_i U^2 Y_i^2\left(\omega\right), \quad i = 1,\ldots,4n \tag{22}$$

The above analysis enables the calculation of the element activity for a given single harmonic excitation. The activity index that is used by MORA is independent of the excitation amplitude, as shown in Eq. (16), and therefore can be set to an arbitrary value, e.g., set to one (1) for simplicity. Model complexity and which physical phenomena need to be included can be determined given the element activity in Eq. (22) and MORA. The complexity of the beam is investigated in the next section in order to identify the significant elements based on beam length and element location. A series of analyses is performed in order to get more insight into the beam dynamics under different scenarios.

4. BEAM COMPLEXITY BASED ON ACTIVITY

The activity metric and MORA is applied to a steel cantilever beam with parameters $\rho = 7860$ kg/m^3, $E = 210$ GPa, $G = 80$ GPa, $A = 3\times10^{-3}$ m^2, $I = 2\times10^{-5}$ m^4, $\kappa = 0.85$. The length of the beam is varied, $L = 0.2$-2 m, in order to study the variation of element significance. The methodology is easy and computationally inexpensive to implement due to the simple and closed form expressions used for calculating the state space matrices, frequency response and activity.

First, the beam length is set to 2 m such that the beam is considered to be slender. The number of segments is set to $n = 30$ and therefore there are a total of 120 energy storage elements modeling the beam. The modeling target is to accurately predict static behavior to low frequency dynamics, thus the excitation frequency is set to 95% of the first natural frequency (122.68 rad/s).

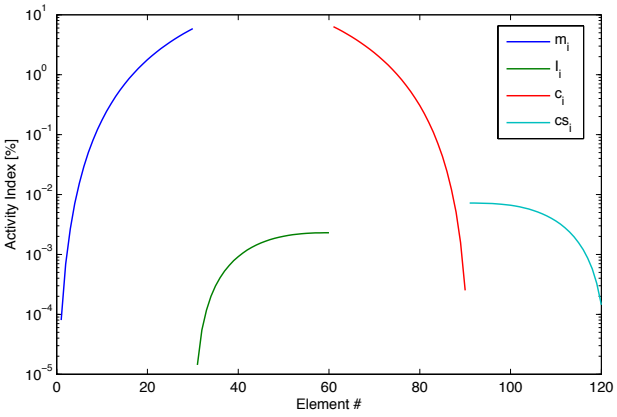

Figure 4: Element activity indices for slender beam.

The results of the activity analysis using Eq. (22) and under these assumptions are shown in Figure 4 where the activity index of all 120 elements is shown. Element numbers 1-30 represent the activity index of the linear inertia (m_i) and 31-60 the activity index of the rotational inertia (I_i) of each segment. Next, element numbers 61-90 and 91-120 represent the activity index of the bending (c_i) and shear (cs_i) compliance, respectively. For each range of elements the smallest numbers represent elements that are next to the fixed end of the beam. It is clear from the activity analysis that the most important elements are related to the linear inertia and the bending stiffness of the beam. On the contrary, the elements

related with the rotary inertia and shear stiffness have very low activity and thus are insignificant under these conditions. The activity analysis agrees with common practice, in which a slender beam is modeled using the Euler-Bernoulli theory that neglects rotational inertia and shear stress effects.

Model complexity is systematically addressed using MORA as it is described in Section 2.3. Elements are ranked according to their activity index as shown in Figure 5 where the sorted activity indices along with the cumulative activity index are plotted. According to activity analysis, 40 of the 120 elements account for almost 99% of the energy the flows through the model. This is a significant result verifying that unnecessary complexity is included in the model, however, the figure does not directly depicts the elements that are insignificant and could be eliminated from the model.

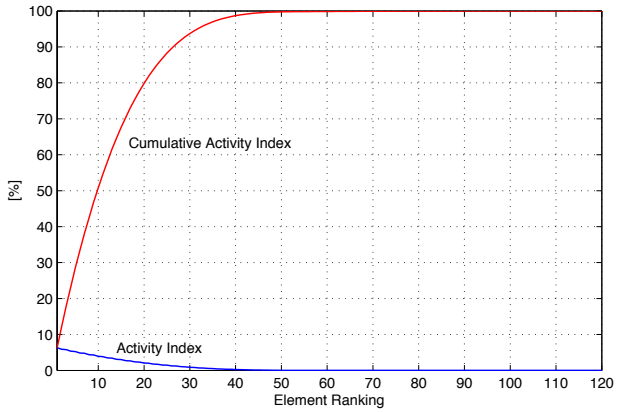

Figure 5: Element ranking for slender beam.

The important elements are next identified using MORA. Using a reduction threshold, $\beta = 99\%$, MORA identifies the elements that have a significant contribution to the system dynamic behavior. The results of this analysis are shown in Figure 6 where both the activity and elimination/inclusion in the reduced model are depicted. The '+' symbol identifies the elements with significant contribution and must be included, where the 'o' symbols identifies that an element is insignificant and must be eliminated from the full model in order to generate the reduced model. Out of the 120 elements only 42 are important and the remaining 78 can be eliminated. More specifically, MORA identifies that all rotational inertia and shear stiffness elements must be eliminated from the model. Linear inertia elements that are close to the support have low activity and can be eliminated from the model, where inertia elements towards the free end of the beam have high activity and must be retained. The reverse is true for the bending stiffness elements, where the elements towards the free end can be eliminated and the ones near the support must be retained. More specifically, 21 of the linear inertia and 21 of the bending stiffness elements have high activity and must be included in the reduced model.

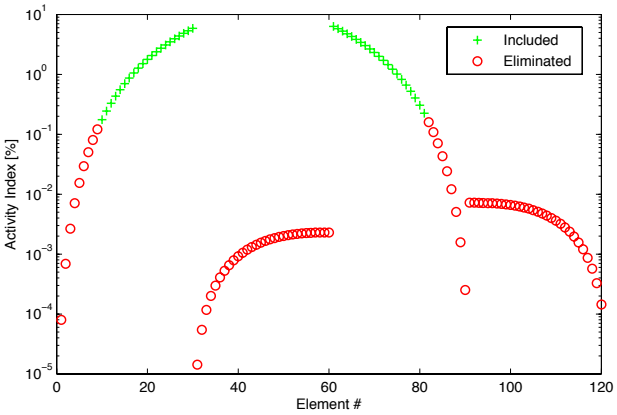

Figure 6: Model reduction for slender beam, $L = 2$ m.

The same reduction using MORA is performed with different beam lengths in order to study how element importance changes as the length is reduced. The reduction for a beam length of 0.7 m is shown in Figure 7. The same trend is observed for the elimination of linear inertia and bending stiffness elements. The activity index of all rotational inertia elements (31-60) is higher than before ($L = 2$ m) but still very low, and therefore, they are eliminated from the model. The activity of shear stiffness (91-120) also increases and some of these elements become important. The shear stiffness elements that are close to the support have higher activity index and have to be included in the reduced model, while the ones towards the free end are eliminated. A total of 59 elements are included in the reduced model with 24 linear inertia, 24 bending stiffness and 11 shear stiffness elements.

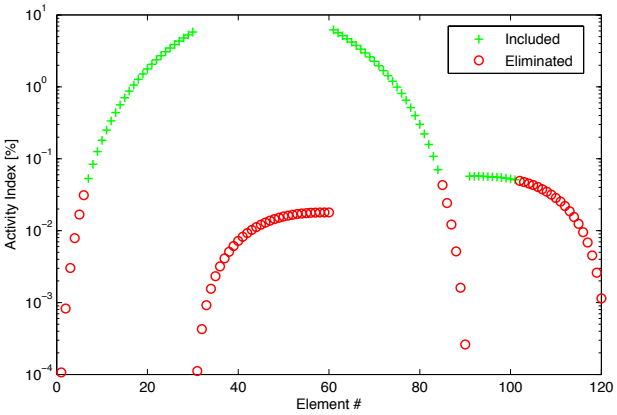

Figure 7: Model reduction for $L = 0.7$ m.

The beam length is further reduced to 0.2 m in order to examine if more elements become important. The activity index of the linear inertia and bending stiffness remains almost unchanged as shown in Figure 8. However, the activity index of the rotational inertia and shear stiffness is further increased such that some of the rotational inertia elements also become important. More specifically the rotational inertia elements that towards the free end are important and the ones near the fixed end are eliminated. A total of 89 elements out of 120 are included in the reduced model with 24 linear inertia, 17 rotational inertia, 23 bending stiffness and 25 shear stiffness elements.

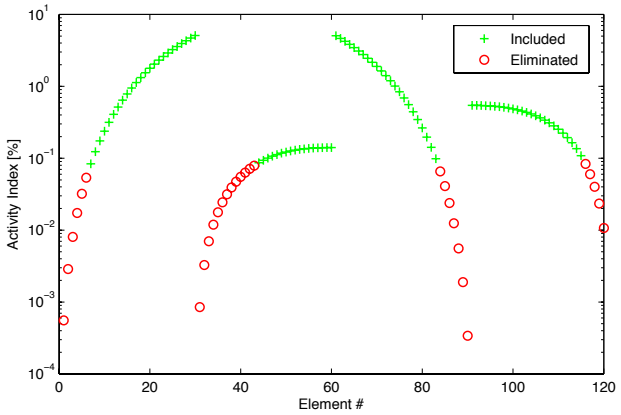

Figure 8: Model reduction for $L = 0.2$ m.

The variation of beam length showed that the total number of important elements increases as the beam length decreases. This variation is investigated in more detail by varying the beam length from 0.2 to 2 m with a step of 20 mm. The number of included linear and rotational inertia, and bending and shear stiffness is recorded along with the total number of elements. The results of this analysis are shown in Figure 9. The total number of elements is monotonically increasing as the beam length is decreased. The number of linear inertia and bending stiffness remains almost constant as the length changes. On the contrary, the number of shear stiffness elements is zero until about 0.9 m where it becomes important and starts increasing. Further reduction in length results in a monotonic increase in the number of included shear stiffness element. A similar behavior is observed for the number of the rotational inertia elements, however, they become important at a lower beam length of about 0.5 m.

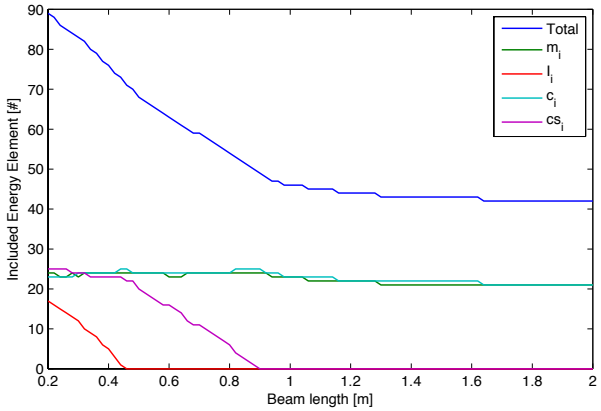

Figure 9: Model reduction for length variation.

5. DISCUSSION AND CONCLUSIONS

A new methodology is developed that reduces the complexity of a Timoshenko or Euler-Bernoulli beam model by providing more insight into the beam dynamic behavior at the same time. The proposed methodology provides a systematic modeling procedure for cantilever beams that are modeled through the finite segment approach. The previously developed activity metric is used as the basis for determining the physical phenomena that need to be included in each segment in order for the model to accurately predict the dynamic behavior of a beam. The procedure starts with the most complicated model, Timoshenko in this case, and then eliminates insignificant elements that do not contribute to the dynamic behavior.

The results presented in this work are in agreement with the assumptions of beam theories, which propose that the Timoshenko beam model must be used for shorter rather than slender beams. The proposed methodology can be used when modeling beams, in order to decide which of the two models to use, Timoshenko or Euler-Bernoulli. In addition, the activity metric can refine the modeling assumptions by identifying what physical phenomena need to be included in each segment, i.e., linear and rotational inertia, bending and shear stiffness.

The number of segments is a significant parameter when it comes to modeling with the finite segment approach but it was considered constant in the analyses of the presented results. The methodology was also performed with various, lower and higher, number of segments, however these results are not presented in this paper for brevity. The reduced models for different number of segments are identical with the ones presented in this work. The only difference is the actual number of included elements, as shown in Figure 9, however, the ratio of included elements to the total number of elements remains constant.

The activity analysis is performed for a given single excitation frequency that is lower than the first natural frequency. This excitation is chosen since the model is expected to be used with low frequency excitations. A similar analysis with the one presented in this work can be performed for a higher frequency or range of frequencies in order to account for more realistic excitations. However, this procedure has to be formalized and this remains as an item for future research.

Because this work uses an energy-based modeling metric, it is convenient to use a model representation and formulation approach from which energy can be easily extracted/calculated. The bond graph approach explicitly presents the power topography of a dynamic system, and therefore, it is used in this work for calculating the necessary variables required for the power calculations. To be clear, the use of this methodology is not limited to systems represented by bond graphs. It can also be applied when the continuous system is modeled using any other modeling methodology, e.g., Lagrange's equations, Newton's Law, etc. However, in this case the calculation of power that is required for the proposed methodology might not be as trivial as using the bond graph formulation.

The results of this paper provide more insight into the nature of the reduced ordered models produced by MORA, and therefore, demonstrate that MORA is an even more useful tool than previously realized for the production of proper models of nonlinear systems. The activity metric effectively addresses the model complexity of distributed parameter components and in addition provides physical insight into the model.

APPENDIX: JUNCTION STRUCTURE MATRICES

$$\mathbf{S} = \left[\begin{array}{c|c|c|c} m_i \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \hline \mathbf{0}_{n\times n} & I_i \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \hline \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & c_i \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ \hline \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & cs_i \mathbf{I}_{n\times n} \end{array}\right]^{-1}$$

$$\mathbf{J}_{SS} = \left[\begin{array}{c|c|c|c} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{J}_1 \\ \hline \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{J}_1 & \Delta x \mathbf{I}_{n\times n} \\ \hline \mathbf{0}_{n\times n} & -\mathbf{J}_1^T & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \hline -\mathbf{J}_1^T & -\Delta x \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{array}\right]$$

$$\mathbf{J}_{SU} = \left\{\begin{array}{c} \mathbf{0}_{(n-1)\times 1} \\ \hline 1 \\ \hline \mathbf{0}_{3n\times 1} \end{array}\right\}$$

$$\mathbf{J}_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}$$

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