A NEW APPROACH TO THE DERIVATION OF A SINGLE SET OF IMPLICIT STATE EQUATIONS FROM A FIXED CAUSALITY BOND GRAPH OF A HYBRID MODEL

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ABSTRACT

This paper proposes a novel approach to the generation of state equations from a bond graph (BG) of a mode switching linear time invariant model. Fast state transitions are modelled by ideal or non-ideal switches. Fixed causalities are assigned following the Standard Causality Assignment Procedure such that the number of storage elements in integral causality is maximised. A system of differential and algebraic equations (DAEs) is derived from the BG that holds for all system modes. It is distinguished between storage elements with mode independent causality and those that change causality due to switch state changes.

A matrix based approach is used for the general case. For illustration, small circuit examples with a special feature are considered. Equations are directly derived from their BG. Switch state commutations may change the index of the DAE system. This is addressed by modelling and simulating one of the examples in the OpenModelica environment.

Keywords: Hybrid models, bond graphs, state equations generation, numerical solution.

1. INTRODUCTION

Various BG representations of hybrid models have been proposed in the literature. A survey may be found in (Borutzky, 2010, Chapter 7). Strömberg et. al. (Strömberg et al, 1993) introduced the ideal switch as another basic BG element excepting that computational causalities at least in some parts of a BG become mode dependent. In the same year, Asher introduced so-called causality resistors (Asher, 1993) which compensate for the causality change at a switch port so that other elements are not affected. In particular, storage elements in preferred integral causality can keep their causality. Disadvantages, however, are that causality resistors can lead to widely different time constants. Their resistance value has to be chosen with care. Buisson et. al. used ideal switches and presented a matrix-based approach to the generation of an implicit state equation from the BG of a mode switching linear time-invariant (LTI) model for a reference mode. Equations for any other system mode are derived from the set of equations for the reference mode (Buisson et al, 2002; Buisson, 1993). However, for systems with n ideal switches there can be quite a number, n_f, of physically feasible switch state combinations $n_f \leq 2^n$ and for each combination a set of implicit state equations must be derived. Moreover, the use of ideal switches means that the preferred integral causality at storage ports is not preserved. The dimension of the state vector becomes mode-dependent. When a closing switch directly connects two storage elements then they become dependent. Their states are algebraically related. As a consequence, simulation software must detect such commutation events, must re-initialise state variables because of a discontinuous jump of their values and continue the computation of the dynamic behaviour with a new set of equations and a smaller number of state variables.

Margetts (Margetts, 2013) also follows the variable causality approach but uses controlled junctions introduced by Mosterman (Mosterman, 1997) for modelling the abstraction of instantaneous discrete switching, marks causalities in the BG that change due to switch state changes by additional dashed causal strokes that indicate the causal configuration after the commutation of some switches and distinguishes between static and dynamic causalities. Other than Buisson et. al. she derives a single set of implicit state equations that holds for all system modes.

In this paper, devices or phenomena with fast state transitions such as electronic diodes and transistors, clutches, or hard mechanical stops are modelled by ideal or non-ideal switches. The unmodified Standard Causality Assignment Procedure (SCAP) is used to once assign *fixed* static causalities so that the number of storage elements in integral causality is maximised. (There may be some storage elements in static derivative causality independent of any switch state changes.) Additional causal strokes are not assigned to bonds. As a result, the BG reflects the configuration only for a specific system mode. Actually, causality at some storage ports may change due to the commutation of some switches. That is, some storage ports change into derivative causality in some system modes, i.e. the number of state variables may be mode-dependent. Therefore, as in (Buisson et al, 2002; Margetts, 2013) it is distinguished between storage elements with mode independent causality and those that change causality due to switch state changes. The latter ones can be identified by inspecting causal paths between storage elements and switches. It is shown that by considering causal paths between storage elements and

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switches a single mode-dependent DAE system can be derived from a BG with static causalities that hold for all system modes.

As in (Buisson et al, 2002), the presented matrix-based equations formulation starts from the well known equivalent block diagram of a general BG extended by a field of switches and a partitioning of the junction structure (JS) matrix. However, other than in (Buisson et al, 2002), a BG, once causally augmented by means of the unmodified SCAP, is used for all modes of operation, i.e. for all (physically feasible) switch state configurations. That is, an actual discrete switch states configuration is not graphically expressed by causal strokes but is taken into account by the values of the discrete switch states. The latter ones are annotated to the switch symbols, i.e. Sw : m. (The non-standard symbol Sw : m denotes a switch that may be either ideal or non-ideal.) This representation may be compared to the *fixed* causality representation of non-ideal switch models composed of a Boolean modulated transformer, MTF : b, and an ON-resistor, R : R_{on}, in fixed conductance causality, in which the value of the transformer modulus, b, accounts for the switch state (Ducreux et al, 1993; Borutzky, 2015).

In this paper, the causality the switches have received by application of the SCAP is disregarded. Instead, the commutation of the switches is always described by an *implicit* equation. Equations obtained from the partitioning of the JS are reformulated so that the outputs of the JS into the switch field can be inserted into the switch equation resulting in an algebraic equation for the outputs of the switches. Its solution is used in the ordinary differential equations (ODEs) for the state variables.

The approach is illustrated by three small circuit examples with each of them having a special feature. Their equations are directly derived from the equivalent BG by following causal paths. Depending on the switch states, the differential equation (ODE) for some storage elements can turn into an algebraic equation which has an affect on the numerical solution of the model. This is addressed in the last section by modelling and simulating one of the examples in the OpenModelica software environment (OpenModelica Consortium, n.d.).

2. EQUATIONS GENERATION

For simplicity it is assumed that the BG of a mode switching LTI model does not contain storage elements that are of static derivative causality independent of any switch states. That is, all storage elements are in integral causality after application of the SCAP. However, there may be causal paths between storage and switch ports which means that a change of a switch state would change the causality of a storage port connected to the switch through a causal path. Let $\boldsymbol{x}_{ii}$ denote the state of all storage elements that are not affected by any switch state changes and remain in integral causality and let $\boldsymbol{x}_{id}$ be the state of all those storage elements in integral causality that would change causality due to a state change of a switch which is not explicitly expressed in the BG as causalities remain fixed once they have been assigned following the SCAP. Nevertheless, the components of the vector $\boldsymbol{x}_{id}$ can be determined by inspecting the BG for causal paths between a storage element in integral causality and a switch. The complementary state variables are grouped into the vectors $\boldsymbol{z}_{ii}$, $\boldsymbol{z}_{id}$ respectively and $\boldsymbol{x}_i = [\boldsymbol{x}_{ii}^T\ \boldsymbol{x}_{id}^T]^T$, $\boldsymbol{z}_i = [\boldsymbol{z}_{ii}^T\ \boldsymbol{z}_{id}^T]^T$.

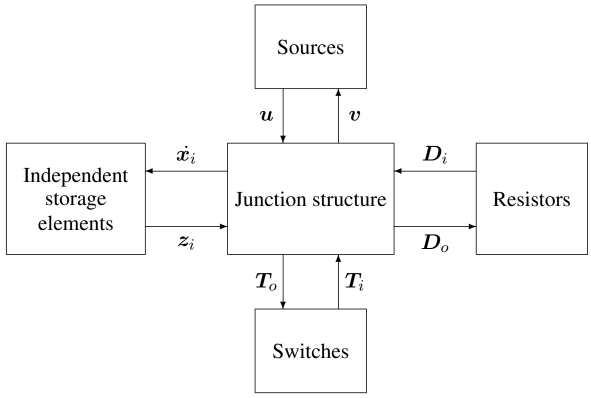

Figure 1: Equivalent block diagram of a general BG with no storage elements in derivative causality

Fig. 1 depicts the well known equivalent block diagram of a general BG with no storage elements in static derivative causality extended by a field of ideal switches.

According to the input and output signals of the junction structure in Fig. 1 the following equations can be set up if it can be assumed that there is no unity gain loop in the junction structure.

$$\underbrace{\begin{bmatrix}\dot{\boldsymbol{x}}_{ii}\\ \dot{\boldsymbol{x}}_{id}\end{bmatrix}}_{\dot{\boldsymbol{x}}_i} = \underbrace{\begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ -\mathbf{S}_{12}^T & \mathbf{0}\end{bmatrix}}_{\mathbf{A}_{11}} \underbrace{\begin{bmatrix}\boldsymbol{z}_{ii}\\ \boldsymbol{z}_{id}\end{bmatrix}}_{\boldsymbol{z}_i} + \underbrace{\begin{bmatrix}\mathbf{S}_{13}\\ \mathbf{S}_{23}\end{bmatrix}}_{\mathbf{A}_{12}} \boldsymbol{D}_i + \underbrace{\begin{bmatrix}\mathbf{S}_{14}\\ \mathbf{S}_{24}\end{bmatrix}}_{\mathbf{A}_{13}} \boldsymbol{T}_i + \underbrace{\begin{bmatrix}\mathbf{S}_{15}\\ \mathbf{S}_{25}\end{bmatrix}}_{\mathbf{A}_{14}} \boldsymbol{u} \tag{1}$$

$$\boldsymbol{D}_o = \underbrace{\begin{bmatrix}-\mathbf{S}_{13}^T & -\mathbf{S}_{23}^T\end{bmatrix}}_{\mathbf{A}_{21}} \begin{bmatrix}\boldsymbol{z}_{ii}\\ \boldsymbol{z}_{id}\end{bmatrix} + \mathbf{S}_{33}\boldsymbol{D}_i + \mathbf{S}_{34}\boldsymbol{T}_i + \mathbf{S}_{35}\boldsymbol{u} \tag{2}$$

$$\boldsymbol{T}_o = \underbrace{\begin{bmatrix}-\mathbf{S}_{14}^T & -\mathbf{S}_{24}^T\end{bmatrix}}_{\mathbf{A}_{31}} \begin{bmatrix}\boldsymbol{z}_{ii}\\ \boldsymbol{z}_{id}\end{bmatrix} + \mathbf{S}_{43}\boldsymbol{D}_i + \mathbf{S}_{44}\boldsymbol{T}_i + \mathbf{S}_{45}\boldsymbol{u} \tag{3}$$

The constitutive equation of the resistive field reads

$$\boldsymbol{D}_i = \mathbf{L}\boldsymbol{D}_o \tag{4}$$

Substituting (4) into (2) gives

$$(\mathbf{I} - \mathbf{S}_{33}\mathbf{L})\boldsymbol{D}_o = \mathbf{A}_{21}\boldsymbol{z}_i + \mathbf{S}_{34}^T\boldsymbol{T}_i + \mathbf{S}_{35}\boldsymbol{u} \tag{5}$$

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The commutations of all n_s ideal switches is expressed by the *implicit* equation

$$\mathbf{M}\boldsymbol{T}_o + \bar{\mathbf{M}}\boldsymbol{T}_i = \mathbf{0} \tag{6}$$

where $\mathbf{M}$ is a diagonal matrix with entries $m_{jj} \in \{1,0\}$, $j = 1,\ldots,n_s$ and $\bar{\mathbf{M}} := \mathbf{I} - \mathbf{M}$. The assignment of fixed static causalities for all elements in a BG of a hybrid model including the switches and (6) are a key point in the presented approach.

Substituting (5) into (4) and the result into (3) assuming that $\mathbf{H} := \mathbf{L}(\mathbf{I} - \mathbf{S}_{33}\mathbf{L})^{-1}$ is non-singular and replacing $\boldsymbol{T}_o$ in the switch equations (6) yields an implicit algebraic equation for $\boldsymbol{T}_i$.

$$\begin{aligned}
\underbrace{\left[\bar{\mathbf{M}} + \mathbf{M}(\mathbf{S}_{44} + \mathbf{S}_{43}\mathbf{H}\mathbf{S}_{34})\right]}_{\mathbf{N}_1}\boldsymbol{T}_i = \\
-\underbrace{\left[\mathbf{M}(\mathbf{A}_{31} + \mathbf{S}_{34}^T\mathbf{H}\mathbf{A}_{21})\right]}_{\mathbf{N}_2}\boldsymbol{z}_i \\
-\underbrace{\left[\mathbf{M}(\mathbf{S}_{45} + \mathbf{S}_{43}\mathbf{H}\mathbf{S}_{35})\right]}_{\mathbf{N}_3}\boldsymbol{u}
\end{aligned} \tag{7}$$

Replacing $\boldsymbol{D}_i$ in (1) gives

$$\begin{aligned}
\dot{\boldsymbol{x}}_i = {} & (\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H}\mathbf{A}_{21})\boldsymbol{z}_i + (\mathbf{A}_{13} + \mathbf{A}_{12}\mathbf{H}\mathbf{S}_{34})\boldsymbol{T}_i \\
& + (\mathbf{A}_{14} + \mathbf{A}_{12}\mathbf{H}\mathbf{S}_{35})\boldsymbol{u}
\end{aligned} \tag{8}$$

Equations 8, 7 constitute a unique DAE system for all modes of operation derived from a BG with fixed static causalities. The complementary state vector $\boldsymbol{z}_i$ can be replaced in (7), (8) by the constitutive equation of the storage field.

$$\begin{bmatrix} \boldsymbol{z}_{ii} \\ \boldsymbol{z}_{id} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{12}^T & \mathbf{F}_{22} \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} \boldsymbol{x}_{ii} \\ \boldsymbol{x}_{id} \end{bmatrix} \tag{9}$$

where $\mathbf{F}$ is symmetric positive definite.

For the states of all storage elements with mode independent causality the ODE reads

$$\begin{aligned}
\dot{\boldsymbol{x}}_{ii} = {} & \underbrace{(\mathbf{S}_{11} - \mathbf{S}_{13}\mathbf{H}\mathbf{S}_{13}^T)}_{\mathbf{A}_1}\boldsymbol{z}_{ii} + \underbrace{(\mathbf{S}_{12} - \mathbf{S}_{13}\mathbf{H}\mathbf{S}_{23}^T)}_{\mathbf{A}_2}\boldsymbol{z}_{id} \\
& + \underbrace{(\mathbf{S}_{14} + \mathbf{S}_{13}\mathbf{H}\mathbf{S}_{34})}_{\mathbf{A}_3}\boldsymbol{T}_i + \underbrace{(\mathbf{S}_{15} + \mathbf{S}_{13}\mathbf{H}\mathbf{S}_{35})}_{\mathbf{A}_4}\boldsymbol{u}
\end{aligned} \tag{10}$$

Substituting (5) into (4) and the result into (1) gives another equation for $\boldsymbol{T}_i$.

$$\begin{aligned}
\underbrace{(\mathbf{S}_{24} + \mathbf{S}_{23}\mathbf{H}\mathbf{S}_{34})}_{:=\mathbf{K}_1}\boldsymbol{T}_i = {} & \dot{\boldsymbol{x}}_{id} + \underbrace{(\mathbf{S}_{12}^T + \mathbf{S}_{23}\mathbf{H}\mathbf{S}_{13}^T)}_{:=\mathbf{K}_2}\boldsymbol{z}_{ii} \\
& + \underbrace{\mathbf{S}_{23}\mathbf{H}\mathbf{S}_{23}^T}_{:=\mathbf{K}_3}\boldsymbol{z}_{id} \\
& - \underbrace{(\mathbf{S}_{25} + \mathbf{S}_{23}\mathbf{H}\mathbf{S}_{35})}_{:=\mathbf{K}_4}\boldsymbol{u}
\end{aligned} \tag{11}$$

or

$$\mathbf{K}_1\boldsymbol{T}_i = \dot{\boldsymbol{x}}_{id} + \underbrace{[\mathbf{K}_2\ \mathbf{K}_3]}_{\mathbf{K}}\boldsymbol{z}_i - \mathbf{K}_4\boldsymbol{u} \tag{12}$$

Given that $\mathbf{K}_1$ can be assumed to be non-singular then substitution of (12) into (10) results in the first part of a final DAE system.

$$\begin{aligned}
\dot{\boldsymbol{x}}_{ii} - \mathbf{A}_3\mathbf{K}_1^{-1}\dot{\boldsymbol{x}}_{id} = {} & [\mathbf{A}_1 + \mathbf{A}_3\mathbf{K}^{-1}\mathbf{K}_2]\boldsymbol{z}_{ii} \\
& + [\mathbf{A}_2 + \mathbf{A}_3\mathbf{K}^{-1}\mathbf{K}_3]\boldsymbol{z}_{id} \\
& + (\mathbf{A}_4 - \mathbf{A}_3\mathbf{K}^{-1}\mathbf{K}_4)\boldsymbol{u}
\end{aligned} \tag{13}$$

Substituting (12) into (7) gives the second part of the final implicit DAE system

$$\begin{aligned}
\mathbf{N}_1\mathbf{K}_1^{-1}\dot{\boldsymbol{x}}_{id} = {} & [\mathbf{N}_2 - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{K}]\boldsymbol{z}_i \\
& - [\mathbf{N}_3 - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{K}_4]\boldsymbol{u}
\end{aligned} \tag{14}$$

Equations 13 and 14 constitute an implicit unique DAE system for all system modes derived from a BG with fixed causalities. Accounting for the constitutive equation of the storage field, (9), the final implicit DAE system is of the form

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A}_3\mathbf{K}_1^{-1} \\ \mathbf{0} & \mathbf{N}_1\mathbf{K}_1^{-1} \end{bmatrix} \underbrace{\begin{bmatrix} \dot{\boldsymbol{x}}_{ii} \\ \dot{\boldsymbol{x}}_{id} \end{bmatrix}}_{\dot{\boldsymbol{x}}_i} = rhs(\boldsymbol{x}_i, \boldsymbol{u}) \tag{15}$$

where *rhs* denotes the right hand side of the DAE system.

In (14), the variables are multiplied by mode dependent matrices. That is, in some cases, the ODE can turn into an algebraic equation, or the right hand side can become zero.

3. ILLUSTRATIVE EXAMPLES

The matrix based equations formulation presented in the previous section can be implemented in a script or coded in a programming language and can be used for an automatic equations generation from BGs of large scale mode switching LTI models. In this section, equations are derived manually from the BGs of three small illustrative example systems by following causal paths and are inserted into the switch equations. For the first two examples with one single switch, the derivation of equations leads to a DAE system with a *mode-dependent index*. In one of the two modes of operation, the system is structurally singular.

3.1 Circuit with a single switch in series with an inductor

As a first example, consider the simple circuit in Fig. 2 with an ideal switch and an inductor in series.

In case the switch is open and fully disconnects inductance L_1 from the two right hand side storage elements, i.e. its OFF conductance is assumed to be zero,

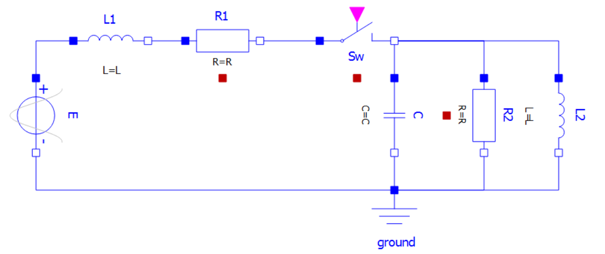

Figure 2: Circuit with an ideal switch and an inductor in series

then the model order is two. In case the switch is closed the model order is three. That is, the number of state variables is mode-dependent. If a non-zero ON resistance of the switch is taken into account and if the switch is modelled as a mode switching resistor with a *fixed* conductance causality then the latter one either causes a causal conflict with the integral causality at the inductor I : L_1 or the inductor would be forced into derivative causality.

The proposed approach adopts an ideal switch model and applies the unmodified SCAP. As a result, all storage elements are in integral causality. By consequence, the switch receives an effort out causality. That is, the causal BG depicted in Fig. 3 reflects the configuration when the switch is closed. The causal path between the switch and the inductor highlighted in red indicates that their causalities change oppositely, i.e. if the open switch state would be taken into account by a flow out causality, then the inductor would be correctly forced into derivative causality.

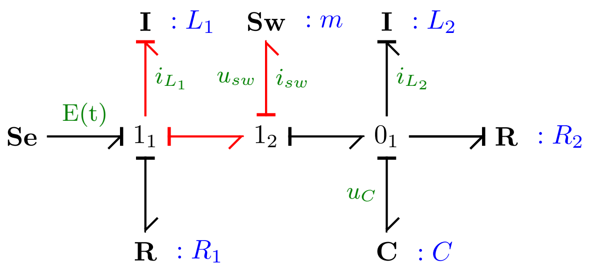

Figure 3: BG of the circuit in Fig. 2

Nevertheless, a single DAE system can be derived from the BG with fixed causalities that holds for both system modes.

$$Sw: \quad 0 = m u_{sw} + \bar{m} i_{sw} \tag{16}$$

$$L_1: \quad L_1 \frac{d}{dt} i_{L_1} = E - R_1 i_{L_1} - u_{sw} - u_C \tag{17}$$

$$C: \quad C \dot{u}_C = i_{sw} - i_{L_2} - \frac{u_C}{R_2} \tag{18}$$

$$L_2: \quad L_2 \frac{d}{dt} i_{L_2} = u_C \tag{19}$$

Only inductor I : L_1 is affected by a causality change at the switch. This is reflected by the causal path between the inductor I : L_1 and the switch. Accordingly, let $\boldsymbol{x}_{ii} := [u_C, i_{L_2}]^T$ and $\boldsymbol{x}_{id} := [i_{L_1}]$. Substituting (17) into the switch equation (16) and accounting for $i_{sw} = i_{L_1} =: i$ yields an implicit DAE that holds for both modes of operation.

$$\left[\begin{array}{cc|c} C & 0 & 0 \\ 0 & L_2 & 0 \\ \hline 0 & 0 & mL_1 \end{array}\right] \frac{d}{dt} \begin{bmatrix} u_c \\ i_{L_2} \\ i \end{bmatrix} = \left[\begin{array}{cc|c} -1/R_2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline -m & 0 & \bar{m} - mR_1 \end{array}\right] \begin{bmatrix} u_c \\ i_{L_2} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m \end{bmatrix} E \tag{20}$$

In (20), the ODE for the inductor current i in the last row turns into the correct algebraic constraint $i = 0$ in case $m = 0$ (open switch).

The ideal switch could also be replaced by a non-ideal switch in fixed conductance causality with an ON resistance R_{on} (forcing inductor $I : L_1$ into derivative causality. Replacing u_{sw} in the new explicit switch equation $i = m u_{sw}/R_{on}$ then gives

$$m L_1 \frac{di}{dt} = -(R_{on} + mR_1) i - m u_C + mE \tag{21}$$

which is also correct for both system modes.

3.2 Circuit with dependent resistors and dependent storage elements

If the switching element in the circuit of Fig. 4 is modelled by an ideal switch then one obtains a higher index problem in case the switch is closed. That is, the index of the DAE system derived from the circuit schematic or from the BG in Fig. 5 is *mode dependent*. The same holds true if a clutch between two inertia is modelled by an ideal switch.

Moreover, as can be seen from the BG in Fig. 5, there is a causal path between the resistors R : R_1 and R : R_2 highlighted in blue indicating that there is an algebraic loop. The latter one can be solved if the constitutive relations of the resistors are linear.

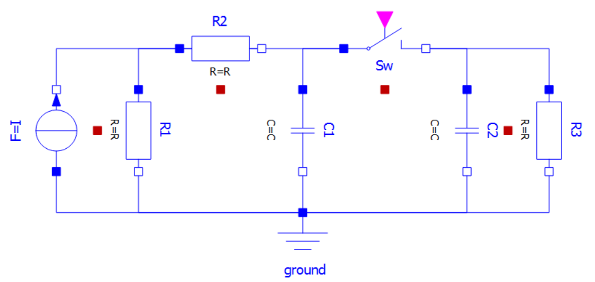

Figure 4: Switched circuit with two dependent resistors and two dependent capacitors

From the BG in Fig. 5, the following equations can be directly derived.

$$u_1 = R_1 (F - i_2) \tag{22}$$

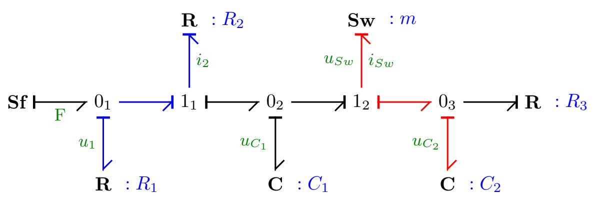

Figure 5: BG of the circuit in Fig. 4

$$
\begin{aligned}
i_2 &= \frac{1}{R_2}(u_1 - u_{C_1}) && (23)\\
C_1\, \dot{u}_{C_1} &= i_2 - i_{\text{Sw}} && (24)\\
C_2\, \dot{u}_{C_2} &= i_{\text{Sw}} - \frac{u_{C_2}}{R_3} && (25)\\
u_{\text{Sw}} &= u_{C_1} - u_{C_2} && (26)\\
0 &= m\, u_{\text{Sw}} + \bar{m} i_{\text{Sw}} && (27)
\end{aligned}
$$

There is another causal path between capacitor $C : C_1$ and the switch and another one between $C : C_2$ and the switch. However, if the switch changes its state, the causality at only one of the two capacitors is affected. The causality at the other capacitor remains fixed. Let $x_{ii} := u_{C_1}$ and $x_{id} := u_{C_2}$. Substituting u_{Sw} and i_{Sw} into the switch equation gives

$$
\left[\begin{array}{c|c} C_1 & C_2 \\ 0 & \bar{m}C_2 \end{array}\right] \left[\begin{array}{c} \dot{u}_{C_1} \\ \dot{u}_{C_2} \end{array}\right] =
\left[\begin{array}{c|c} -1/(R_1+R_2) & -1/R_3 \\ -m & (m-\bar{m}/R_3) \end{array}\right] \left[\begin{array}{c} u_{C_1} \\ u_{C_2} \end{array}\right]
+ \left[\begin{array}{c} R_1/(R_1+R_2) \\ 0 \end{array}\right] F \quad (28)
$$

As in the previous example, the result is an implicit DAE that holds for both system modes. In case the switch is closed ($m = 1$), the ODE for $\dot{u}_{C_2}$ turns into the algebraic equation $u_{C_2} = u_{C_1}$.

3.3 Circuit with two independent switches

The third example illustrates the application of the approach to a simple circuit with two ideal switches that may change their state independently. Fig. 6 shows the circuit diagram.

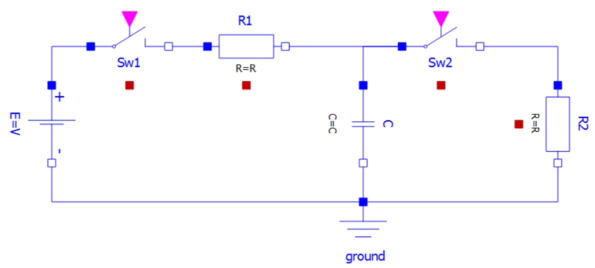

Figure 6: Circuit with two independent switches

From the BG in Fig. 7 it can be seen that there is a causal path from the capacitor $\mathrm{C} : C$ to the switch $\mathrm{Sw} : m_1$. However, the causality of the C element is not affected when the two switches change their state in case linear resistors can be assumed. The path between the C element and the switch $\mathrm{Sw} : m_1$ can be removed by changing the causalities at the switch $\mathrm{Sw} : m_1$ and the resistor $\mathrm{R} : R_1$. The two causal paths from switch $\mathrm{Sw} : m_1$ to resistor $\mathrm{R} : R_1$ and from switch $\mathrm{Sw} : m_2$ to resistor $\mathrm{R} : R_2$ highlighted in blue indicate that the elements in each of the two disjunct paths are algebraically dependent.

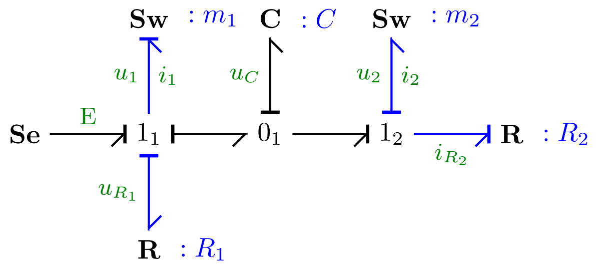

Figure 7: BG of the circuit in Fig. 6

From the BG in Fig. 7, the following equations can be directly derived.

$$
\begin{aligned}
\dot{q} = C\dot{u}_C &= i_1 - i_2 && (29)\\
u_1 &= E - R_1 i_1 - u_C && (30)\\
0 &= m_1 u_1 + \bar{m}_1 i_1 && (31)\\
i_2 &= \frac{1}{R_2}(u_C - u_2) && (32)\\
0 &= m_2 u_2 + \bar{m}_2 i_2 && (33)
\end{aligned}
$$

According to the causal paths between a resistor and a switch in the BG, (30), (31) can be solved for i_1 and (32), (33) yields i_2. Hence,

$$
\dot{q} = \frac{m_1}{m_1 R_1 - \bar{m}_1}(E - u_C) - \frac{m_2}{m_2 R_1 - \bar{m}_2} u_C \quad (34)
$$

which holds for all four system modes. In case both switches are open ($m_1 = m_2 = 0$), (34) correctly yields $\dot{q} = 0$.

4. SOLUTION OF THE DERIVED EQUATIONS

Once a single DAE system for all modes of operation has been derived from a BG of a hybrid model, the question is how the equations can be solved numerically. This problem is discussed by considering the circuit in Fig. 4. It is assumed that the switch abruptly closed at $t_0 = 2\,ms$ and opens abruptly again at $t_1 = 4\,ms$, i.e. $m = 1\ \forall\, t \in [2ms, 4ms]$.

4.1 Direct formulation of the derived equation in Modelica

If the equations 28 are directly formulated in the Modelica language, then the OpenModelica software will notice that there are as many equations as unknowns, will solve

Table 1: Parameters of the circuit in Fig 4

```
record parameters "Parameters IdealSwCircuit"
  //
  parameter Real C1 = 1e-6 "Farad";
  parameter Real C2 = 1e-6 "Farad";
  parameter Real R1 = 1e+3 "Ohms";
  parameter Real R2 = 1e+3 "Ohms";
  parameter Real R3 = 1e+3 "Ohms";
  // source:
  parameter Real tstart1 = 0.0;
  parameter Real h1 = 5e-3 "Amps";
  // trigger:
  parameter Real tstart2 = 0.002;
  parameter Real tstop = 0.004;
  parameter Real eps = 1e-10;
  parameter Real h2 = 1.0 - eps;
end parameters;
```

symbolically the second equation for $\dot{u}_{C2}$ and compile the result. The simulation, however, will fail at $t_0 = 2\,ms$ with a *division by zero*. When the switch closes ($\bar{m} = 0$) the ODE for $\dot{u}_{C2}$ turns into the algebraic constraint $u_{C_2} = u_{C_1}$. The new DAE system for $t \in [2ms, 4ms]$ then needs index reduction. However, symbolic index reduction by means of an implementation of the Pantelides algorithm requires that the index of a DAE system does not change (Cellier and Krebs, 2007). Index reduction is not done *at runtime*. OpenModelica assumes that a hybrid DAE representation is *structurally time invariant*. That is, the set of variables and the set of equations remain fixed during a simulation. Discrete switch states, however, can turn some equations off or on so that the number of *active* equations can vary at run-time (Fritzson, 2004). (For a discussion on hybrid models and their numerical solution see also (Borutzky, 2016, Chap. 5))

4.2 One set of ODEs for each system mode

One pragmatic way to overcome this problem of a mode-dependent DAE index in OpenModelica is to differentiate the algebraic constraint in the case the ideal switch closes ($\bar{m} = 0$) and directly connects the two capacitor voltages so that a set of two ODEs can be provided for each of the two modes.

In addition, both state variables must be initialised at the switching instant $t_0 = 2\,ms$ with a joint value to which both capacitor voltages jump when the switch closes (cf. Lines 16 and 17 of the listing in Fig. 8). This value can be obtained by accounting for *charge conservation*. Let q denote the charge of the capacitors, $u_{C_1}(t_0^-)$ the left side limit and $u_{C_1}(t_0^+)$ the right side limit of $u_{C_1}(t)$.

$$q = C_1 u_{C_1}(t_0^-) \;=\; (C_1 + C_2) u_{C_1}(t_0^+) \qquad (35)$$

When the switch is open, only capacitor C_1 is charged. Solving the ODE for u_{C_1} and using the parameters in Table 1 gives $u_{C_1}(t_0^-) = 5 \cdot (1 - e^{-1}) = 3.1606$.

The switching event at $t_1 = 4\,ms$ does not require additional action as both voltages start from the same value.

```
1  model Circ2
2    ...
3  equation
4    ...
5    //
6    if m < 1 then
7      C1 * der(uC1) + C2 * der(uC2) = (-uC1 / (R1 + R2))
8      - uC2 / R3 + R1 * F / (R1 + R2);
9      C2 * der(uC2) = -uC2 / R3;
10   else
11     C1 * der(uC1) + C2 * der(uC2) = (-uC1 / (R1 + R2))
12     - uC2 / R3 + R1 * F / (R1 + R2);
13     der(uC2) = der(uC1);
14   end if;
15   when pre(m) < m then
16     reinit(uC1, 1.5803);
17     reinit(uC2, 1.5803);
18   end when;
19 end Circ2;
```

Figure 8: Formulation of equations 28 in Modelica

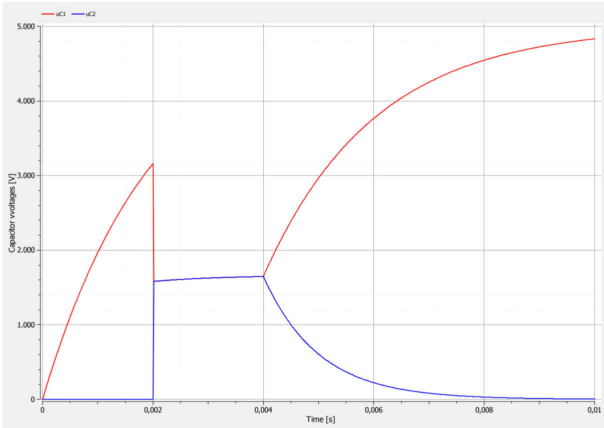

Figure 9: Voltages $u_{C_1}(t)$, $u_{C_2}(t)$ in the circuit of Fig 4 with the switch closed for $t \in [2ms, 4ms]$

Fig. 8 shows part of the Modelica formulation that is accepted by the OpenModelica compiler and for which a simulation provides the correct simulation results displayed in Fig. 9.

4.3 Escape to a non-ideal mode-dependent switch

A DAE system that can be directly formulated in Modelica and solved without problem in OpenModelica can be obtained if the plus sign in the implicit equation of an ideal switch is replaced by a minus sign.

$$0 \;=\; m\, u_{\mathrm{Sw}} - \bar{m}\, i_{\mathrm{Sw}} \qquad (36)$$

with $m \in \{\varepsilon, 1 - \varepsilon\}$ and $0 < \varepsilon \ll 1$.

As a result, in the implicit DAE system, (28), only the coefficients in the second row of the system matrix change their signs. The physical meaning of this modification is that the ideal switch connecting the two parallel capacitors have been replaced by a *non-ideal* switch with a mode-dependent resistance $R_{\mathrm{Sw}}(m) := (1 - m)/m$ that is very large when the switch is open and very small when it is closed. Advantages are that the equations obtained can be directly formulated in Modelica. There is no need to determine the joint value to which both voltages instantaneously jump when the switch is closed in order to re-initialise the numerical integration. Simulation results are the same as in Fig. 9. For this simple, but typical example, neither a loss of accuracy nor a loss of performance is noticeable. For larger models, non-ideal switches with a large OFF-resistance and a small ON-resistance may lead to small time constants and to a longer computational run time. However, in general, not all ideal switches need to be replaced by non-ideal ones.

A similar problem arises with the series connection of an inductor and an ideal switch as in the first example circuit (Fig. 2). One way to overcome the problem in this case is to add a resistor of high resistance between the connection node of inductor and switch and ground. Another option is to replace the ideal switch in series with an inductor by a non-ideal one. As a result, in configurations as the two ones considered in Section 3.1 and 3.2, ideal switches should be replaced by on-ideal ones.

In (Cellier and Kofman, 2006), equations are directly read from the circuit schematic of a half-way rectifier with an inductor and a diode in series as an example of a model with variable structure. The relations between variables and equations are analysed by means of a digraph and it is shown that the equations cannot be solved in case the diode is modelled as an ideal switch because a division by zero occurs when the switch opens. Furthermore it is shown that the problem can be overcome by turning the differential equation of the inductor into an algebraic equation by means of an integration formula. This step the authors call inline integration is inspired by the approach implemented in the circuit simulator Spice in which the constitutive equation of a storage element is replaced by an integration formula. The graphical representation is known as the companion model of a storage element.

Approaches to the numerical solution of the generated equations as the two one presented in Sections 4.2 and 4.3 are necessary because software such as OpenModelica currently can perform an index reduction for a given mode-independent DAE system in a symbolic preprocessing phase prior to compilation and subsequent simulation but not during simulation when an instantaneous mode change results in a change of the index. The problem of handling mode-dependent DAE systems of varying index described in Modelica has been discussed in (Mattsson et al (2015); Elmqvist et al (2014)). First approaches have been implemented in a prototype of the commercial modelling and simulation environment Dymola (Dassault Systèmes (2015)). Also, recently, Zimmer has extended concepts of the Modelica language and has developed an experimental DAE processor that can handle changes in the set of equations at run time (Zimmer, 2013).

CONCLUSION

The paper starts from a BG with ideal or non-ideal switches of mode switching LTI models to which *fixed* causalities have been assigned and presents an approach to the generation of a single set of implicit state space equations that hold for all modes of operation. The novelty of the approach is that no additional causal strokes, or no derivation of equations for a new switch state configuration from the equations of a reference configuration is needed. Dynamic equations for a specific switch state configuration are obtained by just setting the values of the discrete switch states in the mode-dependent state equations accordingly. That is, several switches may commutate simultaneously. The procedure for the generation of the single implicit state equations with coefficient depending on the discrete switch states can be implemented in a script or in a programming language.

In some modes of operation, when closing ideal switches directly connect storage elements so that they become dependent, or when an ideal switch is in series with an inductor, some of the ODEs may turn into algebraic constraints so that index reduction would be required during simulation which, however, is not done by the OpenModelica software. In such cases, OpenModelica cannot solve the mode-dependent equations directly and needs some help from an experienced modeller. It is well known that other software, e.g. Spice for circuit simulation also needs sometimes some small smart model modifications to run a simulation successfully.

The problem of solving mode-dependent DAE Systems of varying index is discussed in the framework of the Modelica language and the OpenModelica software by considering two small example circuits.

The presented approach may be extended by lifting the simplifying assumption that there are no storage elements in derivative causality independent of any switch state changes. Moreover, an extension to mode switching systems with some nonlinear elements may be considered.

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REFERENCES

Asher G (1993) The Robust Modelling of Variable Topology Circuits Using Bond Graphs. In: Granda J, Cellier F (eds) International Conference on Bond Graph Modeling, ICBGM'93, Proc. of the 1993 Western Simulation Multiconference, SCS Publishing, pp 126–131, Simulation Series, volume 25, no. 2, ISBN: 1-56555-019-6

Borutzky W (2010) Bond Graph Methodology – Development and Analysis of Multidisciplinary Dynamic System Models. Springer-Verlag, London, UK, ISBN : 978-1-84882-881-0

Borutzky W (2015) Bond Graph Model-based Fault Diagnosis of Hybrid Systems. Springer International Publishing Switzerland

Borutzky W (ed) (2016) Bond Graphs for Modelling, Control and Fault Diagnosis of Engineering Systems, 2nd edn. Springer International Publishing Switzerland, DOI 10.1007/978-3-319-47434-2

Buisson J (1993) Analysis and Characterisation of Hybrid Systems with Bond Graphs. In: 1993 IEEE International Conference on Systems, Man and Cybernetics, vol 1, pp 264–269

Buisson J, Cormerais H, Richard PY (2002) Analysis of the bond graph model of hybrid physical systems with ideal switches. Proc of the Institution of Mechanical Engineers Part I: Systems and Control Engineering 216(1):47–63

Cellier FE, Kofman E (2006) Continuous System Simulation. Springer-Verlag

Cellier FE, Krebs M (2007) Analysis and Simulation of Variable Structure Systems Using Bond Graphs and Inline Integration. In: Proc. ICBGM 07, 8th SCS Intl. Conf. on Bond Graph Modeling and Simulation, San Diego, CA, USA, pp 29–34

Dassault Systèmes (2015) URL http://www.Dymola.com

Ducreux J, Dauphin-Tanguy G, Rombaut C (1993) Bond Graph Modelling of Commutation Phenomena in Power Electronic Circuits. In: Granda J, Cellier F (eds) International Conference on Bond Graph Modeling, ICBGM'93, Proc. of the 1993 Western Simulation Multiconference, SCS Publishing, San Diego, CA 92177 U.S.A., pp 132–136, Simulation Series, volume 25, no. 2, ISBN: 1-56555-019-6

Elmqvist H, Mattsson S, Otter M (2014) Modelica extensions for Multi-Mode DAE Systems. In: Proceedings of the 10th International Modelica Conference, Lund, Sweden, pp 183 – 193

Fritzson P (2004) Principles of Object-Oriented Modeling and Simulation with Modelica 2.1. Wiley & Sons, ISBN: 0-471-47163-1

Margetts R (2013) Modelling & analysis of hybrid dynamic systems using a bond graph approach. PhD thesis, University of Bath, UK

Mattsson S, Otter M, Elmqvist H (2015) Multi-mode dae systems with varying index. In: Proceedings of the 11th International Modelica Conference, Versailles, France, pp 89 – 98

Mosterman P (1997) Hybrid Dynamic Systems: A hybrid bond graph modeling paradigm and its application in diagnosis. PhD thesis, Vanderbilt University, Nashville, TN, USA

OpenModelica Consortium (n.d.) OpenModelica. URL https://www.openmodelica.org/

Strömberg J, Top J, Södermann U (1993) Variable Causality in Bond Graphs Caused by Discrete Effects. In: Granda J, Cellier F (eds) International Conference on Bond Graph Modeling, ICBGM'93, Proc. of the 1993 Western Simulation Multiconference, SCS Publishing, pp 115–119, Simulation Series, volume 25, no. 2, ISBN: 1-56555-019-6

Zimmer D (2013) A new framework for the simulation of equation-based models with variable structure. SIMULATION 89(8):935–963