A LINEAR PROGRAMMING FORMULATION FOR AN INVENTORY MANAGEMENT DECISION WITH A SERVICE CONSTRAINT

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ABSTRACT

Inventory systems with uncertainty go hand in hand with the determination of a safety stock level. The decision on the safety stock level is based on a performance measure, for example the expected shortage per replenishment period or the probability of a stock-out per replenishment period. The performance measure assumes complete knowledge of the probability distribution during lead time, which might not be available. In case of incomplete information regarding the lead-time distribution of demand, no single figure for the safety stock can de determined in order to satisfy a performance measure. However, an optimisation model may be formulated in order to determine a safety stock level which guarantees the performance measure under the worst case of lead-time demand, of which the distribution is known in an incomplete way. It is shown that this optimisation problem can be formulated as a linear programming problem.

Keywords: inventory management, linear programming, incomplete information

1. INTRODUCTION

Some uncertainty in an inventory system (such as lead time, quantity and quality) depends on the suppliers. If the suppliers introduce too much uncertainty, corrective action should be taken. Some uncertainty, however, is attributable to customers, especially demand. If insufficient inventory is hold, a stock-out may occur leading to shortage costs. Shortage costs are usually high in relation to holding costs. Companies are willing to hold additional inventory, above their forecasted needs, to add a margin of safety.

Determination of an inventory replenishment policy, of the quantities to order, of the review period are typical decisions to be taken by logistics managers. Decisions are made through optimisation models taking a performance measure into consideration which might be cost-oriented or service-oriented. Performance measures of the service-oriented type may be expressed relatively as a probability of a stock-out during a certain replenishment period, or may be expressed absolutely in terms of number of units short, which is a direct indication for lost sales. Both performance measures are taken into consideration and special attention will be paid to feasible combinations of company's objectives regarding both performance measures.

For a definition of both measures we refer to chapter 7 in Silver, Pyke and Peterson (1998) and define the measures as:

The *expected shortage per replenishment cycle* (ESPRC) is defined as (with *t* the amount of safety stock):To avoid any difficulties during the publishing process, authors must not modify any of the styles.

The *expected shortage per replenishment cycle* (ESPRC) is defined as (with *t* the amount of safety stock):

$$ESPRC = \int_{d}^{+\infty} (x-t)\, f(x)\, dx \qquad (1)$$

If ordered per quantity *Q* the fraction backordered is equal to ESPRC/Q and a performance measure, indicated as *P2*, is defined as

$$P_2 = 1 - ESPRC / Q \qquad (2)$$

The other performance measure is the *probability of a stock-out in a replenishment lead time*, defined as:

$$1 - P_1 = \Pr\{x \geq t\} = \int_{d}^{+\infty} f(x)dx \qquad (3)$$

From a production or trading company's point of view, a decision might be formulated to answer the following question: *given a maximum expected number of units short and/or a maximum stock-out probability the company wants to face, what should be the safety inventory at least (or at most)?* The question with the 'at most' option might be only of academic nature, as it reflects the most optimistic viewpoint. In human terms, this question would be interpreted as: 'would there exist any probability distribution so that I can still reach my preset performance criteria given a specific safety inventory?'. This type of question is not relevant for a manager facing a real-life situation.

In case the distribution of demand is known, determining the inventory level, given a maximum shortage or maximum stock-out probability, reduces to the calculation of the inverse cumulative probability function. The decision problem becomes more difficult if incomplete information exists on the distribution of demand during lead time, for example only the range of demand, or the first moment, or the first and second moments are known. In such a case no single value can be determined but rather an interval.

In classical textbooks not too much attention is paid to the shape of the distribution of the demand during lead time. Mostly, based on the first and second moments, the safety stock level is determined using the normal distribution. When of relevance, one rather should look for a distribution, which is defined only for non-negative values and allows for some skewness. In the literature on inventory control, frequent reference is made to the Gamma distribution.

It is generally known that, given a shape of the demand distribution, the higher the coefficient of variation the more a company needs inventory to reach a given service level. In an investigation on the relevance of the demand shape Bartezzaghi, Verganti, and Zotteri (1999) find out that the shape is very relevant. In extreme cases the impact of different demand shapes on inventories is comparable to the effect of doubling the coefficient of variation.

This research deals with the case where the demand distribution during lead time is not completely known. This situation is realistic either with products which have been introduced recently to the market or with slow moving products. In both cases not sufficient data are available to decide on the functional form of the demand distribution function. Some but not complete information might exist like the range of the demand, its expected value, its variance and maybe some knowledge about uni-modality of the distribution.

In case incomplete information is available regarding the demand distribution the integrals of the performance measures P_1 and P_2 cannot be evaluated in an analytical manner. This means that also the inverse problem of determining the safety stock level to satisfy the performance measures cannot be obtained analytically. However, the integrals can be approximated by a linear programming formulation with a large set of constraints.

2. BOUNDS ON THE PERFORMANCE MEASURES IN THE CASE OF INCOMPLETE INFORMATION

In this section the ESPRC measure is focused. First, a link is identified with a similar integral formulation which appears in the field of actuarial sciences. Second, some results, which were obtained in actuarial sciences, are transferred to our type of application.

2.1 Towards an analogy in insurance mathematics

In insurance mathematics, an insurance company using the option of re-insurance is confronted with a stop-loss premium. A stop-loss premium limits the risk X of an insurance company to a certain amount d. If the claim size is higher than d the re-insurance company takes over the risk X-d. The stop-loss premium is based on the expected value of X-d, which in case of a known claim size distribution may be defined as:

$$\int_0^\infty (x-t)_+ \, dF(x) \tag{4}$$

where $F(x)$ represents the claim size distribution (Goovaerts, De Vylder, and Haezendonck 1984).

The same formula (4) may be useful in the performance evaluation of inventory management in case of uncertain demand during lead time. When a company holds t units of a specific product in inventory starting a period between order and delivery, any demand less than t is satisfied while any demand X greater than t results in a shortage of X-t units. A lesser number of units short results in a better service to the customer. In this way formula (4) is a measure for customer service in inventory management.

In the following sections lower and upper bounds are obtained for the performance measure under study, given various levels of information about the demand distribution. From a production or trading company's point of view, a decision might be formulated to answer the following question: *given an expected number of units short the company wants to face, what should be the safety inventory at least or at most?*

2.2 The case of known range, mean and variance

Let the size of the demand X for a specific product in a finite period have a distribution F with first two moments $\mu_1 = E(X)$ and $\mu_2 = E(X^2)$.

From a mathematical point of view, the problem is to find the following bounds:

$$\sup_{F \in f} \int_0^\infty (x-t)_+ \, dF(x) \tag{5a}$$

and

$$\inf_{F \in f} \int_0^\infty (x-t)_+ \, dF(x) \tag{5b}$$

where Ø is the class of all distribution functions F which have moments μ_1 and μ_2, and which have support in R^+. Let further $s^2 = \mu_1 - \mu_2^2$. We assume t to be strictly positive.

For any polynomial $P(x)$ of degree 2 or less, the integral

$$\int_0^\infty P(x) dF(x)$$

only depends on μ_1 and μ_2, so it takes the same value for all distributions in Ø. There exists some distribution G in Ø for which the equality holds:

$$\int_0^\infty P(x) dG(x) = \int_0^\infty (x-t)_+ \, dG(x). \tag{6}$$

As distribution G a two-point or three-point distribution is used. The equality (6) is attained when $P(x)$ and $(x\text{-}t)_+$ are equal in both points of G. The best upper and lower bounds on this term with given moments μ_1 and μ_2 are derived. The method is inspired by papers of Janssen, Haezendock, and Goovaerts (1986) and by Heijnen and Goovaerts (1989). In the following we assume the known range of the distribution to be a finite interval [a,b].

A probability distribution F is called n-atomic if all its probability mass is concentrated in n points at most. The points are called the atoms of the distributions. The problem (5a) has a 2-atomic solution and (5b) has a 3-atomic solution.

If α, β are two different atoms of the 2-atomic probability distribution F satisfying the first-order moment constraint $\int x dF = m_1$, then the corresponding probability masses p_a and p_b are

$$p_a = \frac{m_1 - b}{a - b}, \; p_b = \frac{m_1 - a}{b - a}. \qquad (7)$$

If α, β, γ are three different atoms of the 3-atomic probability distribution F satisfying the moment constraints $\int x dF = m_1, \int x^2 dF = m_2$, then the corresponding probability masses p_a, p_b and p_g are

$$p_a = \frac{s^2 + (m_1 - b)(m_1 - g)}{(a - b)(a - g)},$$
$$p_b = \frac{s^2 + (m_1 - a)(m_1 - g)}{(b - a)(b - g)}, \qquad (8)$$
$$p_g = \frac{s^2 + (m_1 - a)(m_1 - b)}{(g - a)(g - b)}$$

The domain of the parameters is

$$a \le m_1 \le b, \quad 0 \le s^2 \le (m_1 - a)(b - m_1) \; or$$
$$m_1^2 \le m_2 \le m_1(a + b) - ab \; .$$

Further the following abbreviations are used: $s_{mt}^2 = s^2 + (m_1 - t)^2$ and $c = \frac{1}{2}(a + b)$. Further let m_1 and m_2 be chosen that the previous inequalities hold, then let $r' = \frac{m_2 - m_1 r}{m_1 - r}$ for every $r \in [a, b]$ and $r \ne m_1$.

Before moving towards the application, it should be stated that the bounds and their use in applications can be translated from any distribution defined on $[a,b]$ into the bounds with a distribution defined on $[0,b_0]$, where $b_0 = b - a$. Further let $t_0 = t - a$, $m_{10} = m_1 - a$ and $m_{20} = m_2 - 2am_{10} - a^2$. In the following paragraphs we work, without loss of generalisation, with distributions defined on $[0,b_0]$.

The use of the bounds is illustrated by means of a numerical example. Let the demand be defined on the interval [25,75]. The demand follows a distribution with only the following characteristics known: $\mu_1 = 45$ and $\mu_2 = 975$. This means that in Tables 1 and 2, the following values have to be used for $m_{10} = 20$, $m_{20} = 600$, $b_0 = 50$, $b_0' = 13.333$, and $0' = 30$. The values for upper and lower bounds are shown in tables 3 and 4. From Tables 3 and 4, a decision-maker may be decide which level of inventory to hold, given a target value on the number of units short W as a performance measure. From these tables he can derive upper bounds on t_0, which correspond to a pessimistic viewpoint and lower bounds on t_0, which correspond to an optimistic viewpoint. The values corresponding to both viewpoints for the numerical example under study are given in tables 5 and 6.

Table 1: Lower Bounds on the Stop-loss Premium in an interval $[0,b_0]$

Lower bounds	**Conditions**
$m_{10} - t_0$	$0 \le t_0 \le b_0'$ or $0 \le t_0 \le (m_{20} - m_{10}b_0)/(m_{10} - b_0)$
$(m_{20} - m_{10}t_0)/b_0$	$b_0' \le t_0 \le 0'$ or $(m_{20} - m_{10}b_0)/(m_{10} - b_0) \le t_0 \le m_{20}/m_{10}$
0	$0' \le t_0 \le b_0$ or $m_{20}/m_{10} \le t_0 \le b_0$

Table 2: Upper Bounds on the Stop-loss Premium in an interval $[0,b_0]$

Upper bounds	**Conditions**
$m_{10}(m_{20} - m_{10}t_0)/m_{20}$	$t_0 \le 0'/2$ or $t_0 \le m_{20}/(2m_{10})$
$(m_{10} - t_0 + \sqrt{(m_{20} - m_{10}^2) + (t_0 - m_{10})})/2$	$0'/2 \le t_0 \le (b_0 + b_0')/2$ or $m_{20}/(2m_{10}) \le t_0 \le (m_{20} - b_0^2)/(2(m_{10} - b_0))$
$(m_{20} - m_{10}^2)(b_0 - t_0)/((m_{20} - m_{10}^2) + (b_0 - m_{10})^2)$	$(b_0 + b_0')/2 \le t_0$ or $(m_{20} - b_0^2)/(2(m_{10} - b_0)) \le t_0$

Table 3: Lower Bounds on the Number of Units Short for the illustrative example

Lower bounds	Conditions
$20 - t_0$	$0 \le t_0 \le 13.333$
$12 - 2/5\, t_0$	$13.333 \le t_0 \le 30$
0	$30 \le t_0 \le 50$

Table 4: Upper Bounds on the Number of Units Short for the illustrative examp le

Upper bounds	Conditions
$20 - 2/3\, t_0$	$0 \le t_0 \le 15$
$(20 - t_0 + \sqrt{200 + (t_0 - 20)})/2$	$15 \le t_0 \le 31.667$
$(100 - 2t_0)/11$	$31.667 \le t_0 \le 50$

Table 5: Lower bounds on the safety inventory level

Lower bounds	Requirements
$30 - 5/2\, W$	$W \le 6.667$
$20 - W$	$6.667 \le W$

Table 6: Upper bounds on the safety inventory level

Upper bounds	Requirements
$50 - 11/2\, W$	$W \le 3.333$
$(50 - W^2 + 20W)/W$	$3.333 \le W \le 10$
$(60 - 3W)/2$	$10 \le W$

3. A METHOD TO DETERMINE SAFETY STOCK IN THE CASE OF INCOMPLETE INFORMATION ON DEMAND

It has been shown in Janssens and Ramaekers (2008) how the optimisation problem (5a) with constraints interms of first and second moment of the demand distributions, has a dual program which is a linear program with an infinite number of constraints. In Goovaerts, Haezendonck, and De Vylder (1982) an idea is launched to replace the set of constraints by a large finite subset and then to solve the so obtained linear program.

The method assumes that integral constraints can be transformed into a sequence, with increasing number of evaluation points, of optimisation problems and where the integral is replaced by an infinite sum. Instead of evaluating the objective function on a continuous interval [low,high], the functions are evaluated in a discrete number of points x_i ($i = 1..N$). The assumption reflects the idea that if $N \to \infty$ the solution of the continuous problem is found.

This leads to an optimisation problem, where:
t = the level of the safety inventory
p_i = the probability mass in point x_i
z_1 = the expected value of X
z_2 = the absolute second moment of X
z_3 = the maximum allowed number of items short.
The optimisation problem might be formulated as:
[P1] Min t
Subject to

$$\sum_i p_i = 1$$

$$\sum_i x_i p_i = z_1$$

$$\sum_i x^2{}_i p_i = z_2$$

$$\sum_i (x_i - t)_+ p_i \le z_3$$

where $(x_i\text{-}t)_+$ stands for max(x_i-t,0). The decision variables in [P1] are t and p_i (i= 1..N), where N represents the number of discrete points which have been chosen in the experiment.

Problem [P1] offers the answer to the following question: what is the minimal amount of inventory so that a distribution with given characteristics exists in which the expected number short maximally equals the value z_3.

The non-linear constraint may be approximated by letting the value of t coincide with one of the x_i –values (so as $N \to \infty$, the approximation takes the correct value). In such a way the constraint in linearised.

In the case t coincides with a point x_j then

$$\sum_{i=1}^{n} p_i (x_i - x_j)_+ \le z_3 \quad .$$

A binary variable needs to be introduced to indicate the condition ‘t = x_j’. In the case t does not coincide with a point x_j, a general truth should be indicated, for example, ‘the expected number short cannot be larger than the expected demand’, expressed by a binary variable y_j.

$$y_j = 1 \; if \quad t = x_j \\ \quad else \quad 0 .$$

As t can coincide with only one x_j-value, the additional constraint is introduced:

$$\sum_{j=1}^{n} y_j = 1.$$

The *y*-variable is introduced in the last constraint as:

$$\sum_{i=1}^{n} p_i (x_i - x_j)_+ \leq z_3 y_j + z_1(1 - y_j)$$

Finally a link should be made between *t* and the value of *x* with which *t* coincides

$$t \geq x_i y_i, \forall i$$

If *y* = 0, a universal truth is mentioned.

This elaboration will be illustrated by means of the example used in the previous section. With b_0 = 50, the first and second moments in the interval [0,50], the following values μ_{10} = 20 and μ_{20} = 600 are used. The worked out example, in LINDO code, is shown in figure 1. In this approximation 10 intervals of equal length in the interval [0,50] are chosen. Inclusion of both boundaries of the interval, the linear program makes use of 11 x_i-variables.

Take for example the maximum number of units short *W* = 6. From table 5, it can be obtained that the lower bound for *t* equals *t* = 15. The linear program in figure 1 leads to a minimum of *t* = 15, with probability mass in three evaluation points x_1 (*X*=0), x_4 (*X*=15) and x_{11} (*X*=50). The respective probability masses are: p_1 = 0.06667, p_4 = 0.76195 and p_{11}= 0.17143.

```
min t
subject to
p1 + p2 + p3 + p4 + p5 + p6 + p7 + p8 + p9 + p10 + p11 = 1
0 p1 + 5 p2 + 10 p3 + 15 p4 + 20 p5 + 25 p6 + 30 p7 + 35 p8 + 40 p9 + 45 p10 + 50 p11 = 20
0 p1 + 25 p2 + 100 p3 + 225 p4 + 400 p5 + 625 p6 + 900 p7 + 1225 p8 + 1600 p9 + 2025 p10 + 2500 p11 = 600
5 p2 + 10 p3 + 15 p4 + 20 p5 + 25 p6 + 30 p7 + 35 p8 + 40 p9 + 45 p10 + 50 p11 + 14 y1 < 20
5 p3 + 10 p4+ 15 p5 + 20 p6 + 25 p7 + 30 p8 + 35 p9 + 40 p10 + 45 p11 + 14 y2 < 20
5 p4 + 10 p5 + 15 p6 + 20 p7 + 25 p8 + 30 p9 + 35 p10 + 40 p11 + 14 y3 < 20
5 p5 + 10 p6 + 15 p7 + 20 p8 + 25 p9 + 30 p10 + 35 p11 + 14 y4 < 20
5 p6 + 10 p7 + 15 p8 + 20 p9 + 25 p10 + 30 p11 + 14 y5 < 20
5 p7 + 10 p8 + 15 p9 + 20 p10 + 25 p11 + 14 y6 < 20
5 p8 + 10 p9 + 15 p10 + 20 p11 + 14 y7 < 20
5 p9 + 10 p10 + 15 p11 + 14 y8 < 20
5 p10 + 10 p11 + 14 y9 < 20
5 p11 + 14 y10 < 20
14 y11 < 20
1 t > 0
1 t - 5 y2 > 0
1 t - 10 y3 > 0
1 t - 15 y4 > 0
1 t - 20 y5 > 0
1 t - 25 y6 > 0
1 t - 30 y7 > 0
1 t - 35 y8 > 0
1 t - 40 y9 > 0
1 t - 45 y10 > 0
1 t - 50 y11 > 0
y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 + y9 + y10 + y11 = 1
end
int y1 .. int y10
```

Figure 1: LINDO Code for the Illustrative Example

4. AN APPLICATION IN THE SINGLE-PERIOD (NEWSVENDOR) INVENTORY PROBLEM

The single period-inventory problem or newsvendor problem aims to decide the stock quantity of an item when there is a single purchasing opportunity before the start of the selling period and the demand for the item is unknown. A trade-off exists between the risk of overstocking (forcing disposal below the unit purchasing cost) and the risk of understocking (losing the opportunity of making a profit) (Gallego and Moon 1993). Many extensions to the newsvendor problem have been proposed in the last decades, including dealing with different objectives and utility functions, different supplier pricing policies, different news-vendor pricing policies and discounting structures, different states of information about demand, constrained multi-products, multiple-products with substitution, random yields, and multi-location models (Khouja 1999).

Assume a single product is to be ordered at the beginning of a period and can only be used to satisfy demand in that period. The relevant costs on basis of the ending inventory are:

```
min 0.35 Q + 0.9 w1 + 0.9 w2 + 0.9 w3 + 0.9 w4 + 0.9 w5 + 0.9 w6 + 0.9 w7 + 0.9 w8 + 0.9 w9 + 0.9 w10
subject to
p1 + p2 + p3 + p4 + p5 + p6 + p7 + p8 + p9 + p10 + p11 = 1
0 p1 + 5 p2 + 10 p3 + 15 p4 + 20 p5 + 25 p6 + 30 p7 + 35 p8 + 40 p9 + 45 p10 + 50 p11 = 20
0 p1 + 25 p2 + 100 p3 + 225 p4 + 400 p5 + 625 p6 + 900 p7 + 1225 p8 + 1600 p9 + 2025 p10 + 2500 p11 = 600
5 p2 + 10 p3 + 15 p4 + 20 p5 + 25 p6 + 30 p7 + 35 p8 + 40 p9 + 45 p10 + 50 p11 - w1 + 1000 y1 < 1000
5 p3 + 10 p4 + 15 p5 + 20 p6 + 25 p7 + 30 p8 + 35 p9 + 40 p10 + 45 p11 - w2 + 1000 y2 < 1000
5 p4 + 10 p5 + 15 p6 + 20 p7 + 25 p8 + 30 p9 + 35 p10 + 40 p11 - w3 + 1000 y3 < 1000
5 p5 + 10 p6 + 15 p7 + 20 p8 + 25 p9 + 30 p10 + 35 p11 - w4 + 1000 y4 < 1000
5 p6 + 10 p7 + 15 p8 + 20 p9 + 25 p10 + 30 p11 - w5 + 1000 y5 < 1000
5 p7 + 10 p8 + 15 p9 + 20 p10 + 25 p11 - w6 + 1000 y6 < 1000
5 p8 + 10 p9 + 15 p10 + 20 p11 - w7 + 1000 y7 < 1000
5 p9 + 10 p10 + 15 p11 - w8 + 1000 y8 < 1000
5 p10 + 10 p11 - w9 + 1000 y9 < 1000
5 p11 - w10 + 1000 y10 < 1000
1 Q - 0 y1 > 0
1 Q - 5 y2 > 0
1 Q - 10 y3 > 0
1 Q - 15 y4 > 0
1 Q - 20 y5 > 0
1 Q - 25 y6 > 0
1 Q - 30 y7 > 0
1 Q - 35 y8 > 0
1 Q - 40 y9 > 0
1 Q - 45 y10 > 0
1 Q - 50 y11 > 0
y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 + y9 + y10 + y11 = 1
end
int y1 .. int y11
```

Figure 2: LINDO Code of the Newsvendor Example

c_0 = cost per unit of positive inventory remaining at the end of the period (overage cost)

c_1 = cost per unit of unsatisfied demand (underage cost).

Further let:

Q: order quantity

D: random demand with a distribution F with density f defined on a finite interval $[a,b]$ with $a \geq 0$ and $b > a$.

Define $G(Q,D)$ as the total overage and underage cost incurred at the end of period when Q units are ordered at the start of the period and D is the demand. Then it follows that

$$G(Q,D) = c_o \max(0, Q-D) + c_u \max(0, D-Q) \quad (9)$$

The expected cost G(Q) = E[G(Q,D)] can be calculated as:

$$G(Q) = c_0 \int_0^Q (Q-x) f(x) dx + c_u \int_Q^\infty (x-Q) f(x) dx \quad (10)$$

(Nahmias 1993).

The newsvendor formulation also can be used to make decisions in a profit framework. This formulation needs information about the unit cost c, a mark-up m indicating the relative return per currency unit sold and a discount d indicating the loss per currency unit unsold (Gallego and Moon 1993) :

c: unit cost ($c > 0$)

p: unit selling price ($p = (1+m)c$, $m > 0$)

s: unit salvage value ($s = (1-d)c$, $d > 0$).

The expected profit in function of the order quantity, $P(Q)$, can be written as:

$$P(Q) = p\, E \min(Q,D) + sE(Q-D)_+ - cQ \quad (11)$$

since $min(Q,D)$ units are sold, $(Q-D)^+$ are salvaged, and Q units are purchased. Gallego and Moon (1993) show that maximizing $P(Q)$ is equivalent to minimizing

$$dQ + (m+d)E(D-Q)_+ \quad (12)$$

Similar to the case in section 3, the non-linear part of the objective function may be approximated by letting the value of Q coincide with one of the x_i –values (so as $N \rightarrow \infty$, the approximation takes the correct value). In such a way the constraint in linearised. The objective function takes the form:

$$dQ + (m+d)\sum_{j=1}^{N} w_j \quad (13)$$

in which the newly introduced variables w_j take the values

$$w_j = \sum_{i=1}^{n} p_i (x_i - x_j)_+$$

in the case Q coincides with a point x_j and 0 otherwise.

This logic can be introduced in some of the constraints making use of a binary variable introduced