FEED FORWARD NEURAL NETWORK AND SIMULATION OF FILTER ADAPTATION BY DAPHNIA

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ABSTRACT

A model of feeding adaptation of a filter feeder is presented. Based on the assumption that filtration adaptability represents optimization type process is incorporated. Two possible strategies were followed: an instantaneous optimality at each time interval and an integral formulation, maximization of the integral biomass which leads to the optimal control problem with control and state constraints. The optimal control problem is transcribed into nonlinear programming problem, which is implemented with adaptive critic feed forward neural network and recurrent neural network for solving nonlinear projection equations. Also stability analysis of equilibria and some numerical simulation is given. It is shown that Hopf bifurcation may occur depending on filtration rate.

Keywords: optimal control problem, adaptive critic neural network, feeding adaptation, Daphnia

1. INTRODUCTION

Optimal control of nonlinear systems is one of the most active subjects in control theory. There is rarely an analytical solution although several numerical computation approaches have been proposed (for example, see (Kirk, 1989) and (Polak, 1997). The most of the literature dealing with numerical methods for the solution of general optimal control problems focuses on algorithms
for solving discretized problems. The basic idea of these methods is to apply nonlinear programming techniques to the resulting finite dimensional optimization problem (Buskens and Maurer, 2000). When *Euler* integration methods are used, the recursive structure of the resulting discrete time dynamic can be exploited in computing first-order necessary condition.

In the recent years the neural networks are used for obtaining numerical solutions to optimal control problem (Padhi, Unnikrishnan, Wang and Balakrishnan, 2001), (Padhi, Balakrishnan and Randoltph, 2006). For the network, a feed forward network with one hidden layer, a steepest descent error back propagation rule, a hyperbolic tangent sigmoid transfer function and a linear transfer function were used.

The paper presented extends adaptive critic neural network architecture proposed by (Padhi, Unnikrishnan, Wang and Balakrishnan, 2001) to the optimal control problems with control and state constraints. The organization of the paper is as follows. In Section 2 optimal control problems with control and state constraints are being introduced. We summarize necessary optimality conditions and give a short overview on basic result including iterative numerical methods and discussed discretization methods for given optimal control problem and a form of resulting *nonlinear programming problems*. Section 3 presented a short description of *adaptive critic neural network synthesis* for optimal problem with state and control constraints. Section 4 consists of the model of feeding adaptation. In Section 5 and 6 we apply the methods to the model presented in section 4 to compare short-term and long-term strategy of feeding adaptation of filter feeders. Section 7 consists of stability analysis of equilibria. Conclusions are being presented in Section 8.

2. OPTIMAL CONTROL PROBLEM

We consider nonlinear control problem subject to control and state constraints. Let $x(t) \in R^n$ denote the state of a system and $u(t) \in R^m$ the control in a given time interval $[t_0, t_f]$.

Optimal control problem is to minimize

$$J(x,u) = g\left(x(t_f)\right) + \int_{t_0}^{t_f} f_0\left(x(t),u(t)\right)dt \qquad (1)$$

subject to

$$\dot{x}(t) = f\left(x(t),u(t)\right),$$

$$x(t_0) = x_0,$$

$$\psi(x(t_f)) = 0,$$

$$c\left(x(t),u(t)\right) \le 0, \ \ t \in [t_0,t_f].$$

The functions $g: R^n \to R$, $f_0: R^{n+m} \to R$, $f: R^{n+m} \to R^n$, $c: R^{n+m} \to R^q$ and $\psi: R^{n+m} \to R^r$, $0 \le r \le n$ are assumed to be sufficiently smooth on appropriate open sets. The theory of necessary conditions for optimal control problem of form (1) is well developed (Kirk, 1989), (Pontryagin, Boltyanskij, Gamkrelidze and Mischenko,1983).

We introduce an additional state variable

$$x_0(t) = \int_0^t f_0\big(x(s), u(s)\big)ds$$

defined by the

$$\dot{x_0}(t) = f_0\big(x(t), u(t)\big), x_0(0) = 0.$$

Then the augmented Hamiltonian function for problem (1) is

$$H(x, \lambda, \mu, u) = \sum_{j=0}^{n} \lambda_j f_j(x, u) + \sum_{j=0}^{q} \mu_j c_j(x, u), \qquad (2)$$

where $\lambda \in R^{n+1}$ is the adjoint variable and $\mu \in R^q$ is a multiplier associated to the inequality constraints. Let $(\hat{x}, \hat{u})$ be an optimal solution for (1) then the necessary condition for (1) (Kirk, 1989), (Pontryagin, Boltyanskij, Gamkrelidze and Mischenko,1983) implies that there exist a piecewise continuous and piecewise continuously differentiable *adjoint function* λ: $[t_0, t_f] \to R^q$, $\mu(t) \geq 0$ and a multiplier $\sigma \in R^r$ satisfying

$$\dot{\lambda_j}(t) = -\frac{\partial H}{\partial x_j}(\hat{x}(t), \lambda(t), \mu(t), \hat{u}(t))$$

$$\lambda_j(t_f) = g_{x_j}\left(\hat{x}(t_f)\right) + \sigma\psi_{x_j}\left(\hat{x}(t_f)\right), j = 0, \dots, n \quad (3)$$

$$\dot{\lambda_0}(t) = 0$$

$$0 = \frac{\partial H}{\partial u}\big(\hat{x}(t), \lambda(t), \mu(t), \hat{u}(t)\big).$$

For free terminal time t_f , an additional condition needs to be satisfied:

$$H(t_f) = (\sum_{j=0}^{n} \lambda_j f_j(x, u) + \sum_{j=0}^{q} \mu_j c_j(x, u))\,|_{t_f} = 0.$$

Furthermore, the complementary conditions hold i.e. in $t \in [t_0, t_f]$, $\mu \geq 0$, $c(x, u) \leq 0$ and $\mu\, c(x, u) = 0$. Herein, the subscript x or u denotes the partial derivative with respect to *x* or *u*.

2.1. Discretization of optimal control problem

Direct optimization methods for solving the optimal control problem are based on a suitable discretization of (1). Choose a natural number N and let $t_i \in [t_0, t_f], i = 1, \dots, N-1$, be an equidistant mesh point with $t_i = t_0 + ih$, $i = 1, \dots, N$, where h is time step and $t_f = Nh + t_0$. Let the vectors $x^i \in R^{n+1}, u^i \in R^m$, $i = 1, \dots, N$, be approximation of state variable and control variable $x(t_i)$, $u(t_i)$, respectively at the mesh point. *Euler's* approximation applied to the differential equations yields

$$x^{i+1} = x^i + hf(x^i, u^i), \qquad i = 0, \dots, N-1.$$

Choosing the optimal variable $z := (x^0, x^1, \dots, x^{N-1}, u^0, \dots, u^{N-1}) \in R^{N_s}, N_s = (n+1+m)N$, the optimal control problem is replaced by the following discretized control problem in the form of nonlinear programming problem with inequality constraints:

$$min\, J(z) = G(x^N),$$

where

$$G(x^N) = g((x_1, \dots, x_n)^N, t_N) + x_0^N, \qquad (4)$$

subject to

$$-x^{i+1} + x^i + hf(x^i, u^i) = 0,$$

$$x^0 = x(t_0),$$

$$\psi(x^N) = 0,$$

$$c(x^i, u^i) \leq 0,\ \ i = 0, \dots, N-1.$$

In a discrete-time formulation we want to find an admissible control which minimizes object function (4). Let us introduce the *Lagrangian function* for the nonlinear optimization problem (4):

$$L(z, \lambda, \sigma, \mu, h) = \sum_{i=0}^{N-1} \lambda^{i+1}(-x^{i+1} + x^i + f(x^i, u^i)) +$$
$$+G(x^N, t_N) + \sum_{i=0}^{N-1} \mu^i c(x^i, u^i) + \sigma\psi(x^N, t_N). \qquad (5)$$

and define *H*(*i*) and Φ as a follows:

$$H(i) = \lambda(i+1)(x^i + hf(x^i, u^i),$$
$$\Phi = G + \sigma\psi.$$

The first order optimality conditions of Karush-Kuhn-Tucker (Polak, 1997) for the problem (4) are:

$$0 = L_{x^i}(s, \lambda, \mu, h)$$
$$= \lambda^{i+1} + h\lambda^{i+1} f_{x^i}(x^i, u^i) - \lambda^i$$
$$+ \mu^i c_{x^i}(x^i, u^i), i = 0, \dots, N-1, \quad (6)$$

$$0 = L_{x^N}(s, \lambda, \mu, h)$$
$$= G_{x^N}(x^N) + \sigma\psi_{x^N}(x^N, t_N) - \lambda^N (7)$$

$$0 = L_{u^i}(s, \lambda, \mu, h) = h\lambda^{i+1} f_{u^i}(x^i, u^i) + \mu^i c_{u^i}(x^i, u^i),$$
$$i = 0, \dots, N-1, \qquad (8)$$

$$0 = L_h(s, \lambda, \mu, h) = \Phi_h + \sum_{i=0}^{N-1} H_h(i) + \sum_{i=0}^{N-1} \mu^i c_h(x^i, u^i), \quad (9)$$

Eq. (6-9) represents the discrete version of necessary condition (3) for optimal control problem (1).

3. ADAPTIVE CRITIC NEURAL NETWORK FOR OPTIMAL CONTROL PROBLEM WITH CONTROL AND STATE CONSTRAINTS AND FREE TERMINAL CONDITION

It is well known that a neural network can be used to approximate smooth time-invariant functions and uniformly time-varying function (Hornik, Stichcombe and White, 1989). Neurons are grouped into distinct layers and interconnected according to a given architecture (Figure 1). Each connection between two neurons has a weight coefficient attached to it. The standard network structure for an approximation function is the multiple-layer perceptron (or feed forward network). The feed forward network often has one or more hidden layers of sigmoid neurons followed by an output layer of linear neurons.

Figure 1 shows a feed forward neural network with n_i inputs nodes one layer of n_{hl} hidden units and n_o output units. Let $in = [in_1, \dots, in_{n_i}]$ and $out = [out_1, \dots, out_{n_o}]$ be the input and output vectors of the network, respectively. Let $V = [v_1, \dots, v_{n_{hl}}]$ be the matrix of synaptic weights between the input nodes and the hidden units, where $v_j = [v_{j0}, v_{j1}, \dots, v_{jn_i}]$ and v_{j0} is the bias of the jth hidden unit.

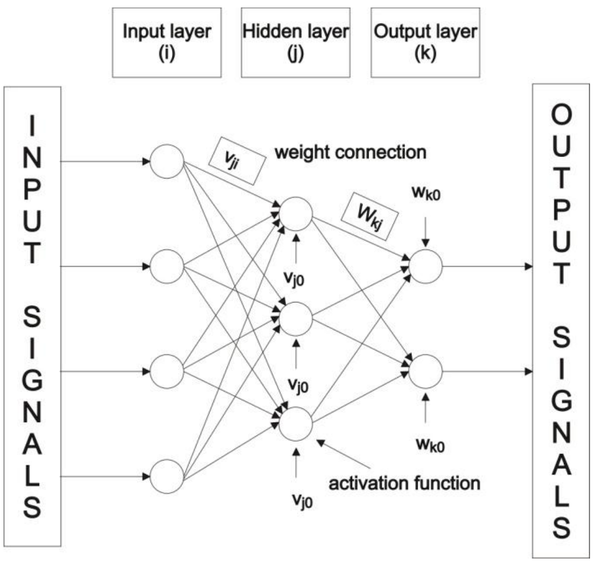

Figure 1: Feed Forward Neural Network Topology With One Hidden Layer, v_{ji}; w_{kj} Are Values of Connection Weights, v_{j0}; w_{k0} Are Values of Bias

Let also $W = [w_1, \dots, w_{n_o}]$ be the matrix of synaptic weights between the hidden and output units, where $w_k = [w_{k0}, w_{k1}, \dots, w_{kn_o}]$ and w_{k0} is the bias of the kth output unit, w_{kj} is the weight that connects the jth hidden units to the kth output unit.
The response of the jth hidden unit is given by

$$hl_j = tanh(\sum_{i=0}^{n_i} v_{ji} in_k),$$

where $tanh(.)$is the activation function for the hidden units. The response of the kth output unit is given by

$$out_k = \sum_{j=0}^{n_{hl}} w_{kj} hl_j.$$

Multiple layers of neurons with nonlinear transfer functions allow the network to learn nonlinear and linear relationships between input and output vectors. The number of neurons in the input and output layers is given, respectively, by the number of input and output variables in the process under investigation.

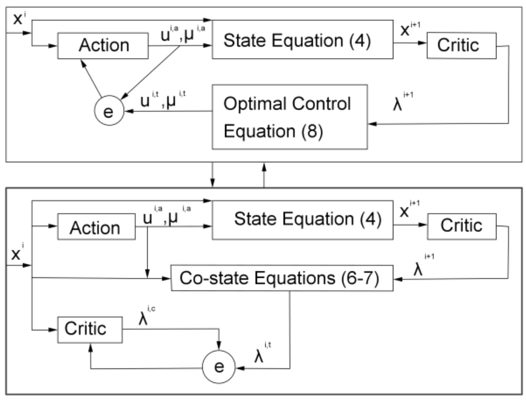

Figure 2: Architecture of Adaptive Critic Network Synthesis

The multi-layered feed forward network shown in Figure 2 is training using the steepest descent error backpropagation rule. Basically, it is a gradient descent, parallel distributed optimization technique to minimise the error between the network and the target output (Rumelhart, Hinton and Wiliams, 1987).

In the Pontryagin's maximum principle for deriving an optimal control law, the interdependence of the state, costate and control dynamics is made clear. Indeed, the optimal control $\hat{u}$ and multiplier $\hat{\mu}$ is given by Eq. (8), while the costate Eqs. (6-7) evolves backward in time and depends on the state and control. The *adaptive critic neural network* is based on this relationship. It consists of two networks at each node: an *action* network the inputs of which are the current states and outputs are the corresponding control $\hat{u}$ and multiplier $\hat{\mu}$, and the *critic* network for which the current states are inputs and current costates are outputs for normalizing the inputs and targets (zero mean and standard deviations). For detail explanation see (Rumelhart, Hinton and Wiliams, 1987).

From free terminal condition ($\psi(x) \equiv 0$) and from Eqs. (6-7) we obtain that $\lambda_0^i = -1$ for $i = N, \ldots, 0$ and $\lambda_j^N = 0$ for $j = 1, \ldots, n$. We use this observation before proceeding to the actual training of the *adaptive critic neural network*. The steps for training the action network are as follows:

1) Generate set S. For all $x^k \in S$, follow the steps below:
(1.i) Input x^k to the action network to obtain $u^{k;a}$ and $\mu^{k;a}$.
(1.ii) Using x^k and $u^{k;a}$ solve state equation (4) to get x^{k+1}.
(1.iii) Input x^{k+1} to the critic network to obtain μ^{k+1}.
(1.iv) Using x^k and μ^{k+1} solve (8) to calculate $u^{k;t}$ and $\mu^{k;t}$.

When

$$\|(u^{k,a}, \mu^{k,a}) - (u^{k,t}, \mu^{k,t})\| \,/\, \|(u^{k,t}, \mu^{k,t})\| < \epsilon_a,$$

the convergence criterion for the action network training is met.

The training procedure for the critic network which expresses the relation between *xk* and *λk* is as follows:

1) Generate set S. For all $x^k \in S$, follow the steps below:
(1.i) Input x^k to the action network to obtain $u^{k;a}$ and $\mu^{k;a}$.
(1.ii) Using x^k and $u^{k;a}$ solve state equation (4) to get x^{k+1}.
(1.iii) Input x^{k+1} to the critic network to obtain λ^{k+1}.
(1.iv) Using x^k, $u^{k,a}$, $\mu^{k,a}$ and λ^{k+1} solve (6) to calculate $\lambda^{k,t}$.
(1.v) Input x^k to the critic network to obtain $\lambda^{k,c}$.

When

$$\|\lambda^{k,c} - \lambda^{k,t})\| \,/\, \|\lambda^{k,t}\| < \epsilon_c,$$

the convergence criterion for the action network training is met.

Further discussion and detail explanation of this adaptive critic method can be found in [6], (Padhi, Unnikrishnan, Wang and Balakrishnan, 2001), (Padhi, Balakrishnan and Randoltph, 2006), (Werbos, 1992), (Kmet, 2011).

4. MODEL OF FEEDING ADAPTATION

The model consists of phosphorus (x_1) as a limiting nutrient for growth of four species of algae of different size ($x_2 - x_5$) and zooplankton (x_6). Similar models of n species of microorganisms competing exploitatively for one, two or more growth-limiting nutrients are used to study continuous culture of microorganisms in chemostat under constant condition (Smith, and Waltman, 1995) without of any predators. Functions occurring in the model are given in Table 2 in ecological and mathematical notation, respectively. The model is described by the following system of ordinary differential equation (9):

$$\dot{x}_1 = a_7(a_8 - x_1) - \sum_{i=2}^{5} \Big(\frac{d_1 x_i p_i x_1}{x_1 + s_i} + r_i f_2 x_i + $$

$$+ x_i x_6 C_i \Big(1 - \frac{d_4}{a_4 + x_i}\Big)\Big) \qquad (10)$$

$$\dot{x}_i = \frac{d_1 x_i p_i x_1}{x_1 + s_i} - r_i f_2 x_i - x_i x_6 E_i - d_2 x_i + a_{i+9} a_7$$

for $i = 2, \ldots, 5$,

$$\dot{x}_6 = x_6\Big(d_3 \sum_{i=2}^{5} \frac{C_i x_i}{a_4 + x_i} - a_5\Big) + a_6$$

Table 1: Values of Parameters

a_1	0.05	sedimentation rate [*day*$^{-1}$]
a_2	0.6	maximum efficiency of zooplankton assimilation
a_3	0.05	recalculation from units of algae to units of zooplankton
a_4	60	half saturation constant for zooplankton feeding [*mg.m*^{3}CHA]
a_5	0.03	zooplankton mortality[*d*$^{-1}$]
a_6	0.002	inflow of zooplankton [*m*$^{-3}$ *C.day*$^{-1}$]
a_7	0.1	hydraulic loading[*d*$^{-1}$]
a_8	200	inflow phosphorus concentration[*mg:m*$^{-3}$*P*]
a_9	0.9	zooplankton filtration rate [*m*$^{-3}$*C.day*$^{-1}$]
a_{10}	120	half saturation constant for light [*cal. cm*$^{-2}$.*day*$^{-1}$]
$a_{11} - a_{14}$	0.	inflow of phytoplankton concentration [*mg.m*$^{-3}$CHA]

For detail explanations see (Kmet and Straskraba, 2004), (Kmet and Kmetova, 2011). It is derived from the models of the series AQUAMOD (Straskraba, and Gnauck, 1985) modified by the inclusion of several "species" of algae. The description of the light dependence of algae is highly simplified. Instead of an approximative integration of the algal growth over depth and time distribution of light intensity only a simple function $g(I)$ is used, describing Michaelis-Menten type dependence with the half saturation constant for light *IKM*. We consider this oversimplification appropriate for the purposes of this paper.

Four species of algae were considered during the computations performed: $x_2, \ldots, x_5$. Each "species" is represented by a particular algal cell (or colony) volume. The volumes were set arbitrarily to (V_i = 50, 500, 2500 and 5000 μm^3), to approximate the set of "edible" algal sizes commonly occurring in our reservoirs. The ecological parameters of the algae are considered functions of V_i (Straskraba, and Gnauck, 1985). Table 1 gives the corresponding values used in the present simulations.

Table 2: Size-specific Parameters of Algae

	V_i	algal cell volume[μm^3)]
	$u_i = 2\sqrt[\frac{1}{9}]{\frac{3V_i}{4\pi}}$	diameter corr. to V_i
	$E_i(u) = \exp(-0.1(u-u_i)^2)$	selectivity
C_i	$Frz(V_i) = a_9 E_i(u)$	forcing function
p_i	$P_{max}(V_i) = 0.5 - 0.05\log V_i$	spec. growth rate[d^{-1}]
r_i	$Resp(V_i) = 0.02 + 0.002\log V_i$	spec. resp. rate of algae [d^{-1}]
s_i	$KS(V_i) = -5 + 10\log V_i$	half sat. constant for P [$mg.m^{-3}P$]
f_i	$Faz = 0.8 + 0.25\cos(t) + 0.12\cos(2t)$	sedimentation func.
f_2	$Temp = 12 + 10\sin(t+220)$	water temperature[^oC]
f_3	$I_0 = 280 + 210\sin(t+240)$	light intensity [$cal.cm^{-2}.day^{-1}$]
	$f(Temp) = e^{0.009Temp}$	$g(I_0) = \frac{I_0}{I_0 + IKM}$
	$d_1 = f.g$ $d_4 = a_2.a_4$	$d_2 = a_1.f_1$ $d_3 = a_3.d_4$

However, for other values of V_i it is possible to derive the parameters from the functions $P_{max}(V_i)$, $KS(V_i)$ and $Resp(V_i)$ given in Table 3. It is to be noted that P_{max} corresponds to light saturation and temperature of 0˚*C*; for 20˚*C* the growth rate will be about 7.2 times higher. The high values of *PRFOS* are used to simulate eutrophic conditions. For the filtration capability of zooplankton we assume that algal volumes selected at a given setting of the filtratory apparatus have log-normal distribution. This is identical with the "size limited predators" and the function we propose is approximately identical with the "selectivity" by this class of predators as given by (Zaret, 1980).

Table 3: Parameters for Four "Species" of Algea

V_i	50	500	2500	5000
u_i	4.57	9.85	16.84	21.22
p_i	0.4151	0.3651	0.3301	0.3151
s_i	11.99	21.99	28.98	31.99
r_i	0.023	0.025	0.027	0.028

The description of selectivity E_i is as follows:

$$E_i(u) = \exp(-0.1(u-u_i)^2)$$

where u is the value of setal density directly related to the algal diameter for which selectivity is maximal and u_i is the diameter corresponding to each algal cell volume V_i. The specific filtration rate of algae of different sizes (volumes) of the population adapted to certain condition (i.e., with certain values of u becomes

$$Frz(V_i) = FRZ * E_i(u),$$

where *FRZ* is the filtration rate for algae of the optimal size, i. e., those which are filtered with the selectivity factors $E_i(u_i) = 1$.

5. OPTIMIZATION

1) instantaneous maximal biomass production as a goal function (local optimality), i.e.,

$$\dot{x}_6 = f_6(x,u,t) \longrightarrow max$$

for all *t*, under the constraints

$$u \in [u_{min}, u_{max}].$$

2) integral maximal biomass (global optimality), i.e.,

$$J(u) = \int_0^T x_6(t)dt,$$

under the constraints

$$u \in [u_{min}, u_{max}].$$

5.1. Local Optimality

In the case of strategy 1, we maximize the following function

$$J(u) = \sum_{i=2}^{5} \frac{E_i(u) d_3 x_i a_9}{(x_i + a_4)}$$

under the constraints

$$u \in [u_{min}, u_{max}].$$

5.2. Global Optimality

In case of strategy 2, we have the following optimal control problem: to find a function

$$J(\hat{u}) = \int_0^T x_6(t)dt$$

attains its maximum, where T denotes the lifetime of an individual Daphnia. We introduce an additional state variable

$$x_0(t) = \int_0^t x_6(s)ds \quad (11)$$

defined by the

$$\dot{x}_0(t) = x_6(t), \qquad x_0(0) = 0.$$

We are led to the following optimal control problems:

$$Maximize\ x_0(t_f) \quad (12)$$

under the constraints

$$c_1(x,u) = u_{min} - u \ \leq 0$$
$$c_2(x,u) = u - u_{max} \ \leq 0.$$

Discretization of Eqs. (10 - 12) using Eqs. (6- 8) and state equation (4) leads to

$$Minimize\ -x_0^N$$

subject to

$$x^{i+1} = x^i + hf(x^i,u^i), \qquad i = 0,\dots,N-1,$$

$$\lambda_i = \lambda^{i+1} + h\lambda^{i+1}f_{x^i}(x^i,u^i) + \mu^i c_{x^i}(x^i,u^i),$$
$$i = N-1,\dots,0,$$

$$\lambda_0^i = -1, \qquad i = 0,\dots,N-1,$$

$$\lambda^N = (-1,0,0,0,0,0,0),$$

$$0 = h\lambda^{i+1}f_{u^i}(x^i,u^i) + \mu^i c_{u^i}(x^i,u^i),$$

where the vector function

$$F(x,u) = (-x_6, f_1(x,u),\dots,f_6(x,u))$$

is given by Eq. (11) and by right-hand side of Eq. (10).

6. NUMERICAL RESULTS

In the adaptive critic synthesis, the critic and action network were selected such that they consist of six and two subnetworks, respectively, each having 6-18-1 structure (i.e. six neurons in the input layer, eighteen neurons in the hidden layer and one neuron in the output layer). The proposed *adaptive critic neural network* is able to meet the convergence tolerance values that we choose, which led to satisfactory simulation results. Simulations, using MATLAB show that proposed neural network is able to solve nonlinear optimal control problem with state and control constraints. Our results are quite similar to those obtained in (Kmet and Straskraba, 2004).

The results of numerical solutions (Figs. 3 - 5) have shown that the optimal strategies $\tilde{u}(t)$ and $\hat{u}(t)$ based on short or long-term perspective, respectively, have different time trajectory for different values of *Faz* - sedimentation function, *Temp* - water temperature, and I_0 - light intensity (t = 120, 210).

Table 4: Results of Goal Function Evaluations for Local and Global Optimality

Value of goal function	t=120	t=210
$local\ J(\tilde{u})$ $global\ J(\hat{u})$	18.4 (Fig. 5) 27.9 (Fig. 5)	105.1 178.4

When $\hat{u}(t)$ is optimal (what is valid according to numerical results) then $J(\hat{u}(t)) \geq J(\tilde{u}(t))$, i.e., the total biomass for the short-term perspective is smaller or maximally equal to the biomass for the long-term perspective. The numerical results have shown, that for the initial conditions considered $J(\hat{u}(t)) > J(\tilde{u}(t))$, (see Table 4). The higher biomass of zooplankton obtained in the case of integral formulation points towards the assumption that the organisms do better if not reacting only to the immediate changes, but having developed mechanisms consistent with more long-term consideration.

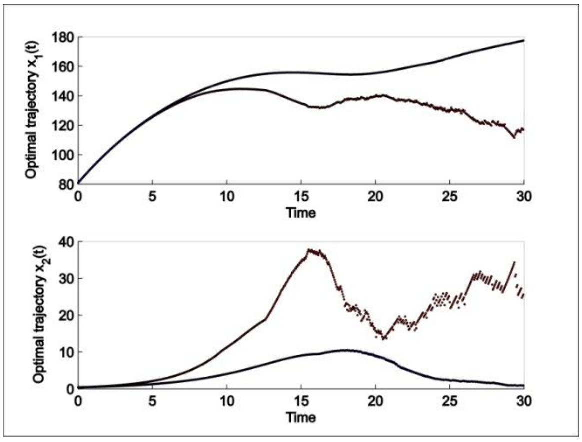

Figure 3: Simulation Results - Local Optimality and Global Optimality ($x_1^0 = 80.4, x_2^0 = 0.4, x_3^0 = 0.3, x_4^0 = 0.2, x_5^0 = 0.2, x_6^0 = 0.1, t = 120$)

7. STABILITY ANALYSIS OF EQUILIBRIA

In this section we investigate the effects of an increasing filter density *u* under constant environmental condition. One parameter analysis of existence and stability of equilibria of (1) is carried out using filter density *u* as a bifurcation parameter. We consider equilibrium solutions to exist only if they lie in the nonnegative cone. Derivatives $\dot{x}_i$ for i = 2,...,5 can be written in the following way

$$\dot{x}_i \ = \ x_i F_i(x,u).$$

Suppose that $x_6 = 0$ and $a_{i+9} = 0$, i = 2, 3, 4, 5. Then we have the following 4 equilibrium points:
$\hat{x}^{i0} = (\hat{x}_{1i}, \hat{x}_2, \dots, \hat{x}_5, 0)$, where

$$\hat{x}_{1i} = \frac{s_i(r_i f_2 + d_2)}{d_1 p_i - r_i f_2 - d_2},$$

$$\hat{x}_i = \frac{a_7(a_8 - x_{1i})}{\frac{d_1 p_1 x_1}{x_1 + s_i} - r_i f_2}$$

for $i = 2, \ldots, 5$,

$\hat{x}_j = 0$ for $j = 2, \ldots, 5, j \neq i$.

Jacobian matrix at equilibrium point $\hat{x}^{20}$ has negative eigenvalues, i.e., it is locally asymptotically stable. The other $\hat{x}^{i0}$, i = 3,4,5 have at least one positive eigenvalue, and they are unstable. Suppose now that x_i = 0 and a_{i+9} = 0, i = 2, 3, 4, 5. Then we have the following equilibrium point

$$\hat{x}^0 = \left(a_8, 0,0,0,0, \frac{a_6}{a_5}\right).$$

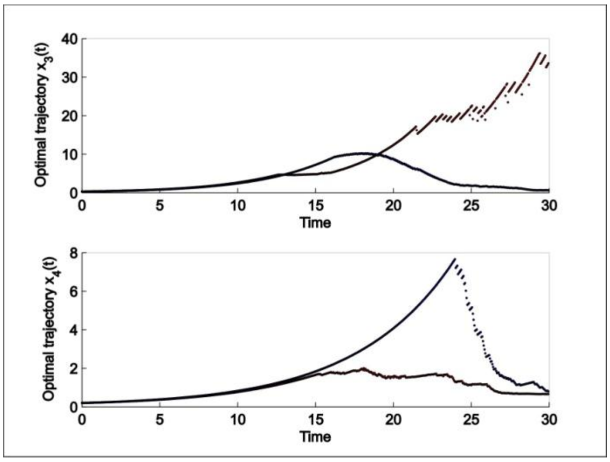

Figure 4: Simulation Results - Local Optimality and Global Optimality ($x_1^0 = 80.4, x_2^0 = 0.4, x_3^0 = 0.3, x_4^0 = 0.2, x_5^0 = 0.2, x_6^0 = 0.1, t = 120$)

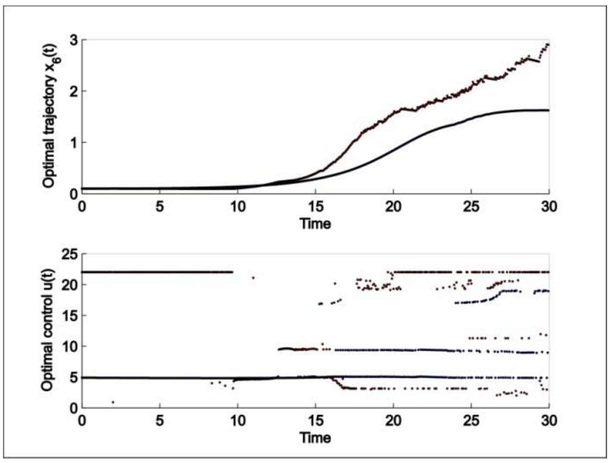

Figure 5: Simulation Results - Local Optimality and Global Optimality ($x_1^0 = 80.4, x_2^0 = 0.4, x_3^0 = 0.3, x_4^0 = 0.2, x_5^0 = 0.2, x_6^0 = 0.1, t = 120$)

The eigenvalues of Jacobian matrix J at $\hat{x}^0$ are:

$\lambda_1 = -a_7,$

$\lambda_i = \frac{d_1 p_i a_8}{s_i + a_8} - r_i f_2 - \frac{a_6}{a_5} E_i - d_2$ for i = 2,3,4,5,

$\lambda_6 = -a_5.$

It follows from the simple calculation that if

$$a_8 > \frac{r_i f_2 + d_2 + \frac{a_6}{a_5} E_i}{d_1 p_i - r_i f_2 - d_2 - \frac{a_6}{a_5} E_i} > 0,$$

then equilibrium point $\hat{x}^0$ is unstable. By similar way as in [3] we can show that if $b_i = d_1 p_i - r_i f_2 - d_2 - \frac{a_6}{a_5} E_i \leq 0$ then

$$\lim_{t \to \infty} x_i(t) = 0.$$

Suppose that b_i > 0 and let us consider the existence of "interior" equilibrium points, where $\hat{x}_i > 0$ for some i = 2, 3, 4, 5. Define

$$R_i = \{x \in R^6;\ x_1, x_i, x_6 \geq 0, x_j = 0 \text{ for } j = 2, \ldots, 5, j \neq i \text{ for } i = 2, \ldots, 5\}$$

$$R_{ik} = \{x \in R^6;\ x_1, x_i, x_k, x_6 \geq 0, x_j = 0 \text{ for } j = 2, \ldots, 5\}$$

for $i = 2, \ldots 5, j = i + 1, \ldots, 5$.

Coordinates $\hat{x}_1, \hat{x}_6$ of equilibrium points are defined by the condition

$$F_2(x,u) = F_3(x,u) = F_4(x,u) = F_5(x,u) = 0.$$

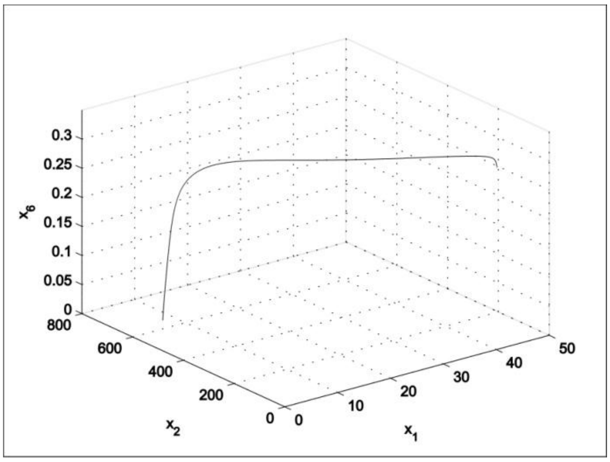

Figure 6: Numerical Solution of System (1) for Initial Condition $x_1^0 = 40.2, x_2^0 = 0.5, x_3^0 = 0.5, x_4^0 = 14.2, x_5^0 = 497.9, x_6^0 = 0.3, u = 12$ and Constant Environmental Condition (t = 120 in Table 1)

For a given set of parameters and function there are ten types of equilibrium points depending on filtration rate u:

$\hat{x}^1 = (\hat{x}_1, \hat{x}_2, 0,0,0, \hat{x}_6) \in R_2$
$\hat{x}^2 = (\hat{x}_1, 0, \hat{x}_3, 0,0, \hat{x}_6) \in R_3$
$\hat{x}^3 = (\hat{x}_1, 0,0, \hat{x}_4, 0, \hat{x}_6) \in R_4$
$\hat{x}^4 = (\hat{x}_1, 0,0,0, \hat{x}_5, \hat{x}_6) \in R_5$
$\hat{x}^5 = (\hat{x}_1, \hat{x}_2, \hat{x}_3 0,0, \hat{x}_6) \in R_{23}$
$\hat{x}^6 = (\hat{x}_1, \hat{x}_2, 0, \hat{x}_4 0, \hat{x}_6) \in R_{24}$
$\hat{x}^7 = (\hat{x}_1, \hat{x}_2, 0,0, \hat{x}_5, \hat{x}_6) \in R_{25}$
$\hat{x}^8 = (\hat{x}_1, 0, \hat{x}_3, 0, \hat{x}_5, \hat{x}_6) \in R_{34}$
$\hat{x}^9 = (\hat{x}_1, 0, \hat{x}_3, 0, \hat{x}_5, \hat{x}_6) \in R_{35}$
$\hat{x}^{10} = (\hat{x}_1, 0,0, \hat{x}_4, \hat{x}_5, \hat{x}_6) \in R_{45}$.

Determining stability of equilibria is accomplished by linearizing the model about steady state and examining the eigenvalues. For the presented model 16 possible kind of equilibria can exist on nonnegative cone.

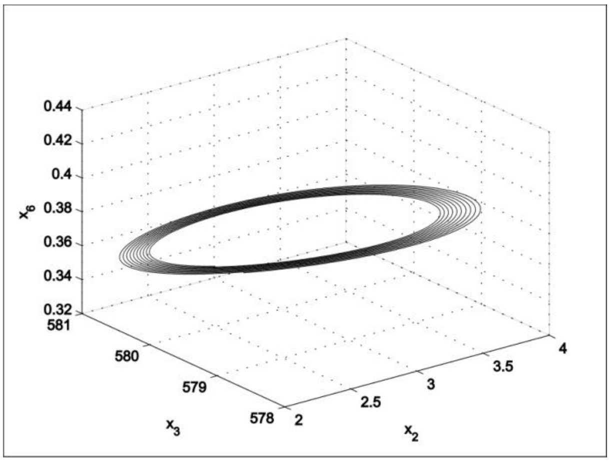

Figure 7: Numerical Solution of System (1) for Initial Condition $x_1^0 = 18.2,\ x_2^0 = 0.8, x_3^0 = 83.1, x_4^0 = 10, x_5^0 = 10, x_6^0 = 0.2, u = 1.5$ and Constant Environmental Condition (t = 120 in Table 1)

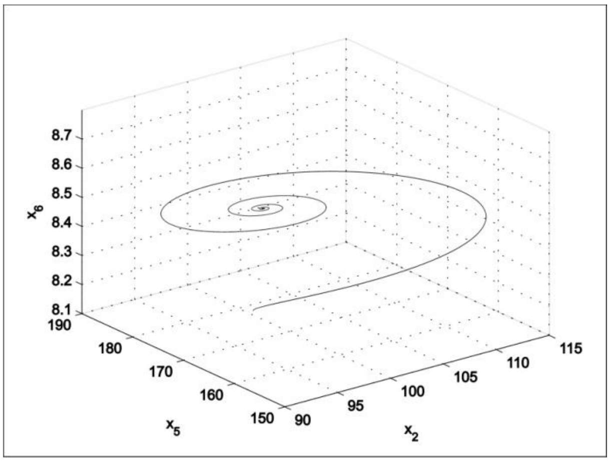

Figure 8: Numerical Solution of System (1) for Initial Condition $x_1^0 = 40.2,\ x_2^0 = 98, x_3^0 = 0.16, x_4^0 = 0.6, x_5^0 = 172.9, x_6^0 = 8.1, u = 10.5$ and Constant Environmental Condition (t = 120 in Table 1)

Jacobian matrix J about equilibria $\hat{x}^5, \hat{x}^6, \hat{x}^7, \hat{x}^8$ and $\hat{x}^{20}$ for different value of u has eigenvalues with negative real part. The simulations seem to indicate that depending on u solution of (1) converges to one of equilibria $\hat{x}^5, \hat{x}^6, \hat{x}^7, \hat{x}^8$ and $\hat{x}^{20}$ or to periodic solution.

Figs. 6, 7 and 8 show the dynamics of algae in a simplified aquatic ecosystem simulating the presence of zooplankton of different body size and correspondingly different filter density u under constant environmental condition (t=120). The comparison of three figures for selected arbitrary constant values of u demonstrates that not only the size but also the number of algal species surviving in the system depends on u. For the environmental conditions specified in the given simulation experiment and u = 12 (Figure 6) the algal sizes x_j for j = 3, 4, 5 converge to zero and only the smallest phytoplankton species x_2 survives and the solution converges to $\hat{x}^{20}$. When u is set to 1.5 or 10.5 two species of algae are able to coexist and the solution converges to periodic orbit or to $\hat{x}^7$, respectively (Figure 7, 8).

With a denser filter, the smaller algae are filtered out more efficiently; because of the nonlinear effects of algal size on ecological parameters, a broader spectrum of species of different sizes is able to survive in the system under the environmental conditions. For the model presented with one growth-limiting nutrient we get that the model exhibits competitive exclusion, only two species of algae are able to survive. Detail analysis of similar systems is given for example in (Kmet and Straskraba, 2004) and (Scheffer, Rinaldi and Kuznetsov, 2000). Numerical solutions shown that in the case of optimal strategies $\hat{u}(t), \tilde{u}(t)$ we have different time trajectories and all species of algae are able to survive (Figure 9, 10).

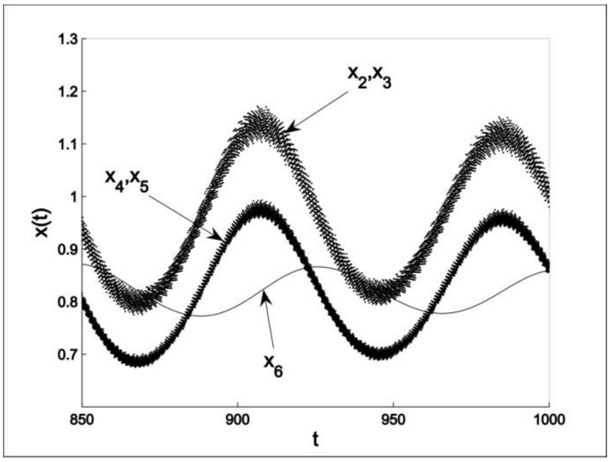

Figure 9: Numerical Solution of System (1) for Initial Condition $x_1^0 = 80.3,\ x_2^0 = 0.4, x_3^0 = 0.3, x_4^0 = 0.2, x_5^0 = 0.1, x_6^0 = 0.1, u$ - Local Optimality Solution and Constant Environmental Condition (t = 120 in Table 1)

8. CONCLUSION

We considered a simple ecological model. One parameter analysis of existence and stability of equilibria was carried out. It is shown that the model has rich dynamics.

Also a single network adaptive critic approach is presented for optimal control synthesis with control and state constraints. We have formulated, analysed and solved an optimal control problem related to the optimal uptake of nutrient by Daphnia. Using MATLAB, a simple simulation model based on adaptive critic neural network was constructed. Numerical simulations have shown that the adaptive critic neural network is able to solve nonlinear optimal control problem with control and state constraints and it explains feeding adaptation of filter feeders of Daphnia.

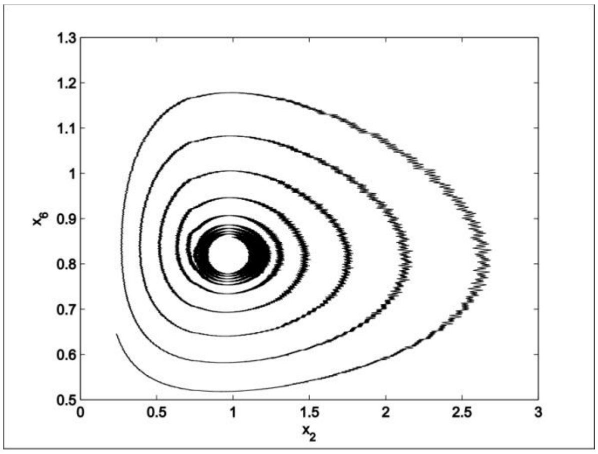

Figure10: Trajectory of System (1) for Initial Condition $x_1^0 = 80.3,\ x_2^0 = 0.4, x_3^0 = 0.3, x_4^0 = 0.2,\ x_5^0 = 0.1,\ x_6^0 = 0.1,\ u$ - Local Optimality Solution and Constant Environmental Condition (*t* = 120 in Table 1)

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